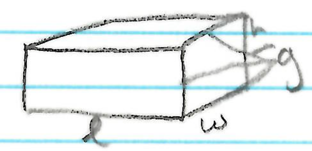
MA 16020

Lesson 24

Relative Extrema (Part 2)

If you are doing an application problem and are given a constraint function which you can easily solve for one variable, solve it for that variable and plug it into the equation you're optimizing.

Ex 1. The post office will accept packages whose combined length and girth is at most 60 inches. What is the largest volume that can be sent in a rectangular box?



girth is the perimeter perpendicular to length.

So $V = lwh$, $g = 2w+2h$, $l+g \leq 60$ Max volume occurs when $l+g=60$, so $l+2w+2h=60$ ← constraint

$l = 60-2w-2h$

$V = (60-2w-2h)wh = 60wh - 2w^2h - 2wh^2$

$V_w = 60h - 4wh - 2h^2$, $V_h = 60w - 2w^2 - 4wh$

$V_w = V_h = 0$ $\quad 60h - 4wh - 2h^2 = 0$ and $60w - 2w^2 - 4wh = 0$

$h(60 - 4w - 2h) = 0$

$h = 0$ or $60 - 4w - 2h = 0 \Rightarrow 2h = 60 - 4w \Rightarrow h = 30 - 2w$

$h = 0$: $60w - 2w^2 = 0 \Rightarrow 2w(30 - w) = 0$, $w = 0$ or $w = 30$

(h, w) are $(0,0)$, $(0,30)$

$h = 30-2w$: $60w - 2w^2 - 4w(30-2w) = 0$

$60w - 2w^2 - 120w + 8w^2 = 0$

$6w^2 - 60w = 0 \Rightarrow 6w(w-10) = 0 \Rightarrow w=0$ or $w=10$

when $w=0$, $h = 30-2(0) = 30$; when $w=10$, $h = 30-2(10) = 10$

$(0,0)$, $(10,10)$

Only sensible possibility is $h=10, w=10$.
(others give volume of 0)!
Can check $h=10, w=10$ is a local max.

$V = 60wh - 2w^2h - 2wh^2$
$= 60(10)(10) - 2(10)^2(10) - 2(10)(10)^2 = 2000$

max volume is $\boxed{2000 \text{ in}^3}$

Ex 2. A shop sells two dresses, dress A which is white and gold and dress B which is blue and black. Each dress costs \$20. If dress A is sold for x dollars and dress B is sold for y dollars, consumers will buy $19 - \frac{3}{2}x + y$ dresses of type A and $2 + 4x - \frac{7}{2}y$ dresses of type B. How should prices be set to maximize profit?

Profit = Revenue − Cost
Revenue = (price of dress A)(# dress A sold) + (price dress B)(# dress B sold)
$= x(19 - \frac{3}{2}x + y) + y(2 + 2x - \frac{7}{2}y)$
$= 19x - \frac{3}{2}x^2 + xy + 2y + 2xy - \frac{7}{2}y^2$
Cost = (cost dress A)(# dress A) + (cost dress B)(# dress B)
$= 20(19 - \frac{3}{2}x + y) + 20(2 + 2x - \frac{7}{2}y)$
$= 20(21 + \frac{1}{2}x - \frac{5}{2}y) = 420 + 10x - 50y$

$P = 19x - \frac{3}{2}x^2 + 3xy + 2y - \frac{7}{2}y^2 - (420 + 10x - 50y)$
$= 9x - \frac{3}{2}x^2 + 3xy + 52y - \frac{7}{2}y^2$

$P_x = 9 - 3x + 3y$, $P_y = 3x + 52 - 7y$
$9 - 3x + 3y = 0$ and $3x + 52 - 7y = 0$

$$\begin{cases} -3x + 3y = -9 \\ 3x - 7y = -52 \end{cases} \rightarrow -4y = -61, \quad y = \frac{61}{4} \approx 15.25$$

$-3x + 3(\frac{61}{4}) = -9 \Rightarrow x = \frac{73}{4} \approx 18.25$

$\boxed{\text{Dress A: \$18.25, Dress B: \$15.25}}$

MA 16020
Lesson 24

Ex 3. A product is being sold at \$200 per unit. If x thousand dollars is spent on development and y thousand dollars is spent on promotion, then

$$\frac{100y}{y+5} + \frac{300x}{x+7} \text{ units will be sold.}$$

How much money should be spent on development and production to maximize profit?

$$\text{Revenue} = 200\left(\frac{100y}{y+5} + \frac{300x}{x+7}\right)$$

$$\text{Cost} = 1000x + 1000y$$

$$P = 200\left(\frac{100y}{y+5} + \frac{300x}{x+7}\right) - 1000x - 1000y$$

$$P_x = 200\left(\frac{(x+7)(300) - (300x)(1)}{(x+7)^2}\right) - 1000 = \frac{420{,}000}{(x+7)^2} - 1000$$

$$P_y = 200\left(\frac{(y+5)(100) - (100y)(1)}{(y+5)^2}\right) - 1000 = \frac{100{,}000}{(y+5)^2} - 1000$$

$$P_x = 0 \Rightarrow \frac{420{,}000}{(x+7)^2} = 1000 \Rightarrow 420 = (x+7)^2 \Rightarrow \sqrt{420} = x+7$$

$$x = \sqrt{420} - 7 \approx 13.494 \text{ so } \boxed{\$13{,}494 \text{ development}}$$

$$P_y = 0 \Rightarrow \frac{100{,}000}{(y+5)^2} = 1000 \Rightarrow 100 = (y+5)^2 \Rightarrow 10 = y+5$$

$$y = 5, \text{ so } \boxed{\$5{,}000 \text{ promotion}}$$

Ex 4. A biologist makes a medium to grow bacteria. The percentage of salt in the medium is given by $S = 0.01x^2y^2z$, where x, y, and z are amounts in liters of 3 different nutrients mixed together to create the medium. The ideal salt percentage is 24%. The costs of x, y, and z are \$5, \$6, and \$7 per liter respectively. Find the minimum cost that can be achieved.

$0.24 = 0.01x^2y^2z \leftarrow$ constraint

$z = \frac{0.24}{0.01x^2y^2} = \frac{24}{x^2y^2}$

cost $= C = 5x + 6y + 7z = 5x + 6y + \frac{168}{x^2y^2}$

$C_x = 5 - \frac{336}{x^3y^2}$, $C_y = 6 - \frac{336}{x^2y^3}$ set both $= 0$

$5 = \frac{336}{x^3y^2} \Rightarrow y^2 = \frac{336}{5x^3} \Rightarrow y = \sqrt{\frac{336}{5x^3}}$

$6 = \frac{336}{x^2\left(\sqrt{\frac{336}{5x^3}}\right)^3} = 336 \cdot \frac{1}{x^2 \cdot \left(\frac{336^{3/2}}{5^{3/2}x^{9/2}}\right)} = \frac{336 \cdot 5^{3/2} \cdot x^{5/2}}{336^{3/2}}$

So $x^{5/2} = \frac{6 \cdot 336^{3/2}}{336 \cdot 5^{3/2}} \Rightarrow x = \left(\frac{6 \cdot 336^{3/2}}{336 \cdot 5^{3/2}}\right)^{2/5} \approx 2.4954$

plug in to $y = \sqrt{\frac{336}{5x^3}} \approx 2.0795$

Now, plug into $C = 5x + 6y + \frac{168}{x^2y^2}$

$\approx \boxed{\$31.19}$