MA 16020
Lesson 25
Lagrange Multipliers (Part 1)

Oftentimes optimization problems come with a constraint. The Method of Lagrange Multipliers gives a way to solve constrained optimization problems.

Method of Lagrange Multipliers

Given a function $f(x,y)$ that you wish to optimize subject to the constraint $g(x,y) = c$ (c is a constant),

1. Find f_x, f_y, g_x, and g_y
2. Introduce the new variable λ with the system of equations
   $f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x,y) = c$
3. Find ordered pairs (a,b) which satisfy all 3 equations at once.
4. Plug all ordered pairs into $f(x,y)$ to see which one optimizes the function.

Warning! : Do not cancel x, y, or λ from any of the equations! Instead factor the equations. Canceling causes a loss of points to check.

Ex 1. Find the minimum value of $f(x,y) = x^2+y^2$ subject to the constraint $\underbrace{4y = 6-3x}_{3x+4y=6}$.

$f(x,y) = x^2+y^2, \quad g(x,y) = 3x+4y, \quad c = 6$

$f_x = 2x, \quad f_y = 2y, \quad g_x = 3, \quad g_y = 4$

$2x = 3\lambda, \quad 2y = 4\lambda, \quad 3x+4y = 6$

$\lambda = \frac{2}{3}x \qquad 2y = 4(\frac{2}{3}x)$, so $y = \frac{4}{3}x$

$3x + 4(\frac{4}{3}x) = 6 \Rightarrow \frac{25}{3}x = 6 \Rightarrow x = \frac{2}{25}$

$y = \frac{4}{3}(\frac{2}{25}) = \frac{8}{75}$

so $f(\frac{2}{25}, \frac{8}{75}) = (\frac{2}{25})^2 + (\frac{8}{75})^2 \approx \boxed{0.0178}$

Ex 2. Find the maximum value of $f(x,y) = e^{9xy}$ subject to the constraint $x^2+y^2=25$. Assume x and y are both positive

$f(x,y) = e^{9xy}$, $g(x,y) = x^2+y^2$

$f_x = 9ye^{9xy}$, $f_y = 9xe^{9xy}$, $g_x = 2x$, $g_y = 2y$

$\underbrace{9ye^{9xy} = \lambda 2x}$, $9xe^{9xy} = \lambda 2y$, $x^2+y^2=25$

$\lambda = \frac{9}{2}\frac{y}{x}e^{9xy}$ $\quad$ $9xe^{9xy} = \left(\frac{9}{2}\frac{y}{x}e^{9xy}\right)2y = 9\frac{y^2}{x}e^{9xy}$

$9xe^{9xy} - 9\frac{y^2}{x}e^{9xy} = 0 \Rightarrow \underbrace{9e^{9xy}}_{\text{Cannot be 0}}\left(x - \frac{y^2}{x}\right) = 0$

$x = \frac{y^2}{x} \Rightarrow x^2 = y^2$

Plugging in to $x^2+y^2=25$, get $y^2+y^2=25$

so $2y^2 = 25 \Rightarrow y^2 = \frac{25}{2}$, so $y = \pm\sqrt{\frac{25}{2}}$

Assume y positive, so $y = \sqrt{\frac{25}{2}}$.

$x^2 = y^2$, so $x^2 = \frac{25}{2} \Rightarrow x = \pm\sqrt{\frac{25}{2}}$, get $x = \sqrt{\frac{25}{2}}$

So max occurs at $\left(\sqrt{\frac{25}{2}}, \sqrt{\frac{25}{2}}\right)$

$f\left(\sqrt{\frac{25}{2}}, \sqrt{\frac{25}{2}}\right) = e^{9\sqrt{\frac{25}{2}}\cdot\sqrt{\frac{25}{2}}} = \boxed{e^{225/2}}$

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Ex 3. Find the maximum value of $f(x,y) = x^2 + 5y^3$ subject to the constraint $4x^2 + y^2 = 100$

$f(x,y) = x^2 + 5y^3$, $g(x,y) = 4x^2 + y^2$

$f_x = 2x$, $f_y = 15y^2$, $g_x = 8x$, $g_y = 2y$

$\underbrace{2x = \lambda 8x}$, $15y^2 = \lambda 2y$, $4x^2 + y^2 = 100$

$2x - \lambda 8x = 0 \Rightarrow 2x(1 - 4\lambda) = 0$

$\underline{x = 0}$ or $1 - 4\lambda = 0 \Rightarrow \lambda = \frac{1}{4}$

$15y^2 = \lambda 2y$, when $\lambda = \frac{1}{4}$, is $15y^2 = \frac{1}{2}y \Rightarrow 15y^2 - \frac{1}{2}y = 0$

$y(15y - \frac{1}{2}) = 0$ so $\underline{y = 0}$ or $\underline{y = \frac{1}{30}}$

Now, plug in to constraint:

$x = 0$: $4(0)^2 + y^2 = 100 \Rightarrow y^2 = 100 \Rightarrow y = \pm 10$

$(0, 10)$, $(0, -10)$

$y = 0$: $4x^2 + (0)^2 = 100 \Rightarrow 4x^2 = 100 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$

$(5, 0)$, $(-5, 0)$

$y = \frac{1}{30}$: $4x^2 + (\frac{1}{30})^2 = 100 \Rightarrow x^2 = 25 - \frac{1}{3600} \Rightarrow x = \pm\sqrt{25 - \frac{1}{3600}}$

$\left(\sqrt{25 - \frac{1}{3600}}, \frac{1}{30}\right)$, $\left(-\sqrt{25 - \frac{1}{3600}}, \frac{1}{30}\right)$

Plug all 6 points into $f(x,y)$

$f(0,10) = (0)^2 + 5(10)^3 = 5000$

$f(0,-10) = (0)^2 + 5(-10)^3 = -5000$

$f(5,0) = (5)^2 + 5(0)^3 = 25$

$f(-5,0) = (-5)^2 + 5(0)^3 = 25$

$f\left(\sqrt{25 - \frac{1}{3600}}, \frac{1}{30}\right) \approx 25$

$f\left(-\sqrt{25 - \frac{1}{3600}}, \frac{1}{30}\right) \approx 25$

So $\boxed{5000}$

Ex 4. The minimum value of $f(x,y)=x^2e^{y^2}$, subject to the constraint $20y^2+x=10$, occurs at what point(s)?

$f(x,y)=x^2e^{y^2}$, $g(x,y)=20y^2+x$

$f_x=2xe^{y^2}$, $f_y=2x^2ye^{y^2}$, $g_x=1$, $g_y=40y$

$2xe^{y^2}=\lambda$, $2x^2ye^{y^2}=\lambda 40y$, $20y^2+x=10$

$2x^2ye^{y^2}=40y(2xe^{y^2})=80xye^{y^2}$

$2x^2ye^{y^2}-80xye^{y^2}=0$

$2xye^{y^2}(x-40)=0$

$x=0$ or $y=0$ or $x=40$

Plug into constraint:

$x=0$: $20y^2+(0)=10 \Rightarrow y^2=\frac{1}{2} \Rightarrow y=\pm\sqrt{\frac{1}{2}}$

$(0,\sqrt{\tfrac{1}{2}})$, $(0,-\sqrt{\tfrac{1}{2}})$

$y=0$: $20(0)^2+x=10 \Rightarrow x=10$

$(10,0)$

$x=40$: $20y^2+(40)=10 \Rightarrow 20y^2=-30 \Rightarrow y^2=-\frac{3}{2}$

impossible! No points.

$f(0,\sqrt{\tfrac{1}{2}})=0e^{1/2}=0$

$f(0,-\sqrt{\tfrac{1}{2}})=0e^{1/2}=0$

$f(10,0)=10e^{(0)^2}=10$

minimum occurs at $\boxed{(0,\sqrt{\tfrac{1}{2}}) \text{ and } (0,-\sqrt{\tfrac{1}{2}})}$