MA 16020
Lesson 26
Lagrange Multipliers (Part 2)

Ex 1. Alice has exactly 24 hours to study for an exam and without preparation, she will get 400 out of 1000 points. Her exam score will improve by $x(50-x)$ points for x hours of reading notes and by $y(60-y)$ points for y hours of review problems. She will lose $(x+y)^2$ points from fatigue. What is the maximum score she can obtain?

$$\text{Points} = P(x,y) = 600 + x(50-x) + y(60-y) - (x+y)^2$$
$$= 400 + 50x - x^2 + 60y - y^2 - (x^2+2xy+y^2)$$
$$P(x,y) = 400 + 50x - 2x^2 + 60y - 2y^2 - 2xy$$

Constraint: $\underbrace{x+y}_{g(x,y)} = \underbrace{24}_{c}$

$P_x = 50-4x-2y$, $P_y = 60-4y-2x$, $g_x = 1$, $g_y = 1$

$50-4x-2y = \lambda$, $60-4y-2x = \lambda$, $x+y=24$

$$50-4x-2y = 60-4y-2x$$
$$-2x+2y = 10, \quad x+y=24$$
$$\begin{cases} -x+y=5 \\ x+y=24 \end{cases} \Rightarrow 2y = 29 \Rightarrow y = \frac{29}{2}$$
$$x = 24 - y = 24 - \frac{29}{2} = \frac{19}{2}$$

$$P\left(\frac{19}{2}, \frac{29}{2}\right) = 400 + 50\left(\frac{19}{2}\right) - 2\left(\frac{19}{2}\right)^2 + 60\left(\frac{29}{2}\right) - 2\left(\frac{29}{2}\right)^2 - 2\left(\frac{19}{2}\right)\left(\frac{29}{2}\right)$$
$$\approx \boxed{869 \text{ points}}$$

Ex 2. A fruit stand sells bananas and strawberries. If x bananas and y strawberries are put on the stand in the morning, there will be a profit of $8x^{3/2}y^{1/2}$ dollars/day. Suppose 150 pieces of fruit can be put on the stand. What is the maximum profit? Round to nearest 100 dollars.

$P(x,y) = 8x^{3/2}y^{1/2}$, $g(x,y) = x+y = 150$

$P_x = 12x^{1/2}y^{1/2}$, $P_y = 4x^{3/2}y^{-1/2}$, $g_x = 1$, $g_y = 1$

$12x^{1/2}y^{1/2} = \lambda$, $4x^{3/2}y^{-1/2} = \lambda$, $x+y = 150$

$12x^{1/2}y^{1/2} = 4x^{3/2}y^{-1/2}$ (multiply by $y^{1/2}$)

$12x^{1/2}y = 4x^{3/2}$

$12x^{1/2}y - 4x^{3/2} = 0$

$4x^{1/2}(3y - x) = 0 \Rightarrow x = 0$ or $x = 3y$

Plug into $g(x,y) = 150$

$\underline{x=0}$: $(0) + y = 150 \Rightarrow y = 150$, $(0, 150)$

$\underline{x=3y}$: $3y + y = 150 \Rightarrow y = \frac{75}{2}$

$x = 3\left(\frac{75}{2}\right) = \frac{225}{2}$ $\left(\frac{225}{2}, \frac{75}{2}\right)$

$P(0, 150) = 0$

$P\left(\frac{225}{2}, \frac{75}{2}\right) = 8\left(\frac{225}{2}\right)^{3/2}\left(\frac{75}{2}\right)^{1/2} \approx \boxed{\$58,500}$

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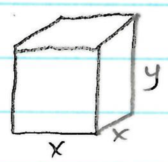
Ex 3. A rectangular box with a square base is to be constructed from material that costs \$10/ft² for the bottom, \$5/ft² for the top, and \$2/ft² for the sides. Find the greatest volume the box can have if it costs \$200. Round to 2 decimals.



$$C = 10(x^2) + 5(x^2) + 2(4xy)$$
$$= 15x^2 + 8xy$$
$$V = x^2 y$$

$V = x^2y$, $\quad C = 15x^2 + 8xy = 200$

$V_x = 2xy$, $V_y = x^2$, $\quad C_x = 30x + 8y$, $C_y = 8x$

$2xy = \lambda(30x + 8y)$, $\underbrace{x^2 = \lambda 8x}$, $15x^2 + 8xy = 200$

$$x^2 - x\lambda 8 = 0$$
$$x(x - \lambda 8) = 0$$
$$x = 0 \text{ or } \lambda = \frac{x}{8}$$

$\underline{x = 0}$: $\quad 2(0)y = \lambda(30(0) + 8y) \Rightarrow 0 = \lambda 8y \Rightarrow y = 0 \quad (0,0)$

$\underline{\lambda = \frac{x}{8}}$: $\quad 2xy = \frac{x}{8}(30x + 8y) = \frac{15}{4}x^2 + xy$

$\Rightarrow 2xy = \frac{15}{4}x^2 + xy \Rightarrow xy - \frac{15}{4}x^2 = 0$

$$x\left(y - \frac{15}{4}x\right) = 0$$

$x = 0$ (already done) or $y = \frac{15}{4}x$

$\underline{y = \frac{15}{4}x}$: $\quad 15x^2 + 8x\left(\frac{15}{4}x\right) = 200 \Rightarrow 45x^2 = 200 \Rightarrow x^2 = \frac{40}{9}$

$x = \pm\sqrt{\frac{40}{9}}$ (negative doesn't make sense)

$y = \frac{15}{4}\left(\sqrt{\frac{40}{9}}\right)$

$V\left(\sqrt{\frac{40}{9}}, \frac{15}{4}\sqrt{\frac{40}{9}}\right) = \frac{40}{9}\left(\frac{15}{4}\sqrt{\frac{40}{9}}\right) \approx \boxed{35.14 \text{ ft}^3}$

$V(0,0) = 0$

Ex 4. On a certain island there are R hundred rats and S hundred snakes. Their populations are related by $(R-20)^2+16(S-15)^2=68$. What is the maximum combined number of snakes and rats that can be on the island? (Round to integer)

$N = 100R + 100S, \quad g(R,S) = (R-20)^2 + 16(S-15)^2 = 68$

$N_R = 100, \quad N_S = 100, \quad g_R = 2(R-20), \quad g_S = 32(S-15)$

$100 = (2R-40)\lambda, \quad 100 = (32S-480)\lambda, \quad (R-20)^2+16(S-15)^2=68$

$\lambda = \frac{100}{2R-40}, \quad \lambda = \frac{100}{32S-480}$

$\Rightarrow \frac{100}{2R-40} = \frac{100}{32S-480} \Rightarrow 100(32S-480) = 100(2R-40)$

$\Rightarrow 16S - 240 = R - 20$

$\Rightarrow R = 16S - 220$

$(16S-220-20)^2 + 16(S-15)^2 = 68$

$(16S-240)^2 + 16(S-15)^2 = 68$

$(16(S-15))^2 + 16(S-15)^2 = 68$

$16^2(S-15)^2 + 16(S-15)^2 = 68$

$(S-15)^2(16^2+16) = 68$

$(S-15)^2 = \frac{68}{16^2+16} = \frac{1}{4}$

$S-15 = \pm\frac{1}{2} \Rightarrow S = 15 \pm \frac{1}{2}$

$R = 16S - 220$

$R = 16(15 \pm \frac{1}{2}) - 220 = \underset{(+)}{28} \text{ or } \underset{(-)}{12}$

$N = 100(15+\frac{1}{2}) + 100(28) = \boxed{4350}$

$N = 100(15-\frac{1}{2}) + 100(12) = 2650$