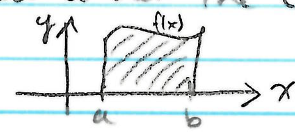
MA 16020

Lesson 27

Double Integrals (Part 2)

For single variable functions $y = f(x)$, $\int_a^b f(x)\,dx$ represents the area under the curve $f(x)$ from $x = a$ to $x = b$.



Since $z = f(x,y)$ is a surface, one might ask if we can find the volume under the surface. We can! We use double integrals.

$\iint_R f(x,y)\,dA$ is the double integral of $f(x,y)$ over the region R in the xy-plane.

Usually it appears as

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx \quad \text{or} \quad \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

We compute double integrals by evaluating the inside integral first, then evaluating the outside integral.

Like in partial derivatives, $\int f(x,y)\,dy$ treats x like a constant, and $\int f(x,y)\,dx$ treats y like a constant.

Ex 1. Compute $\int_0^1 \underbrace{\int_2^3 xy\,dx}_{\text{inside integral}}\,dy$

$$\int_0^1 \tfrac{1}{2}x^2y \Big|_2^3 \,dy = \int_0^1 \left(\tfrac{1}{2}(3)^2y - \tfrac{1}{2}(2)^2y\right) dy$$

$$= \int_0^1 \tfrac{5}{2}y\,dy = \tfrac{5}{4}y^2 \Big|_0^1 = \tfrac{5}{4}(1) - \tfrac{5}{4}(0)$$

$$= \boxed{\tfrac{5}{4}}$$

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Ex 2. $\int_{-2}^{2}\int_{0}^{4}(5x+10y)\,dy\,dx$

$= \int_{-2}^{2}\left(5xy+5y^2\right)\Big|_{y=0}^{y=4}\,dx$

$= \int_{-2}^{2}\left(5x(4)+5(4)^2-5x(0)-5(0)^2\right)dx$

$= \int_{-2}^{2}(20x+80)\,dx$

$= 10x^2+80x\Big|_{x=-2}^{x=2} = 10(2)^2+80(2)-10(-2)^2-80(-2)$

$= \boxed{320}$

Ex 3. $\int_{8\pi}^{10\pi}\int_{y}^{\pi/2} -3\sec(y)\sin(x)\,dx\,dy$

$= \int_{8\pi}^{10\pi} 3\sec(y)\cos(x)\Big|_{x=y}^{x=\pi/2}\,dy$

$= \int_{8\pi}^{10\pi}\left(3\sec(y)\underbrace{\cos(\tfrac{\pi}{2})}_{0} - 3\underbrace{\sec(y)\cos(y)}_{1}\right)dy$

$= \int_{8\pi}^{10\pi} -3\,dy$

$= -3y\Big|_{8\pi}^{10\pi} = -3(10\pi)-(-3(8\pi)) = \boxed{-6\pi}$

Ex 4. $\int_{3}^{5}\int_{1}^{x}\frac{9x^2}{y^2}\,dy\,dx$ (Round to 2 decimals)

$= \int_{3}^{5}\int_{1}^{x} 9x^2y^{-2}\,dy\,dx$

$= \int_{3}^{5} -9x^2y^{-1}\Big|_{1}^{x}\,dx$

$= \int_{3}^{5}\left(-9x^2x^{-1}+9x^2\right)dx$

$= \int_{3}^{5}\left(-9x+9x^2\right)dx$

$= -\frac{9}{2}x^2+3x^3\Big|_{3}^{5}$

$= -\frac{9}{2}(5)^2+3(5)^3-\left(-\frac{9}{2}(3)^2+3(3)^3\right)$

$= \boxed{222.00}$

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Ex 5. $\int_1^e \int_0^{3\ln x} 7x \, dy \, dx$. (Round to 2 decimals)

$= \int_1^e 7xy \Big|_0^{3\ln x} dx$

$= \int_1^e 21x\ln x - 0 \, dx$

Integration by parts

$u = \ln x \qquad v = \frac{21}{2}x^2$

$du = \frac{1}{x}dx \qquad dv = 21x \, dx$

$uv - \int v \, du$

$\frac{21}{2}x^2 \ln x - \int \frac{21}{2} x \, dx$

$= \frac{21}{2}x^2 \ln x - \frac{21}{4}x^2 \Big|_1^e$

$= \frac{21}{2}e^2 - \frac{21}{4}(e)^2 - 0 + \frac{21}{4} \approx \boxed{44.04}$

Ex 6. $\int_0^{\sqrt{\pi/3}} \int_{x^2}^{\pi/2} -2x\sin y \, dy \, dx$

$= \int_0^{\sqrt{\pi/3}} 2x\cos y \Big|_{x^2}^{\pi/2} dx$

$= \int_0^{\sqrt{\pi/3}} (0 - 2x\cos(x^2))dx$

$u = x^2, \quad du = 2x \, dx$

$u(0) = 0^2 = 0, \qquad u(\sqrt{\pi/3}) = \frac{\pi}{3}$

$= \int_0^{\pi/3} -\cos(u) \, du$

$= -\sin(u) \Big|_0^{\pi/3}$

$= -\sin(\frac{\pi}{3}) + \sin(0)$

$= -\frac{\sqrt{3}}{2} + 0$

$= \boxed{-\frac{\sqrt{3}}{2}}$