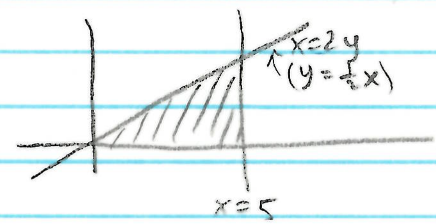
MA 16020
Lesson 29
Double Integrals (Part 3)

Remember some integrals require you to switch the order of integration.

Ex 1. $\int_0^{5/2} \int_{2y}^{5} 3e^{x^2}\,dx\,dy$

The inner integral is impossible.



dx dy:
Region is bounded on left by $x = 2y$
on right by $x = 5$
smallest y value is 0
largest y value is $\frac{5}{2}$

dy dx:
Region is bounded below by $y = 0$
above by $y = \frac{1}{2}x$
smallest x value is 0
largest x value is 5

$$\int_0^5 \int_0^{\frac{1}{2}x} 3e^{x^2}\,dy\,dx$$
$$= \int_0^5 3ye^{x^2}\Big|_0^{x/2}\,dx$$
$$= \int_0^5 \left(\tfrac{3}{2}xe^{x^2} - 0\right)dx$$

$u = x^2$, $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$u(0) = 0^2 = 0$, $u(5) = 5^2 = 25$

$$\int_0^{25} \tfrac{3}{4}e^u\,du$$

$$= \tfrac{3}{4}e^{25} - \tfrac{3}{4}e^0 = \boxed{\tfrac{3}{4}(e^{25} - 1)}$$

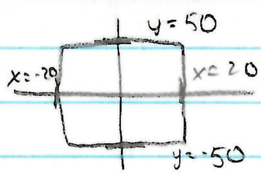
Average Value of $f(x,y)$

Sometimes we want the average value of $f(x,y)$ over a region R. This is given by the formula

$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

where Area(R) is the area of the region R which can sometimes be found using geometry of by using a single variable integral.

Ex 2. A building with a rectangular base has a curved roof whose height is $h(x,y) = 60 - 0.09x^2 + 0.021y^2$. The rectangular base extends from $-20 \le x \le 20$ feet and $-50 \le y \le 50$ feet. Find the average height of the building. (Round to 3 decimals)



$$\frac{1}{A}\int_{-50}^{50}\int_{-20}^{20} (60 - 0.09x^2 + 0.021y^2)\, dx\, dy$$

$$\text{Area} = (100)(40) = 4000$$

$$\text{Ave} = \frac{1}{4000}\int_{-50}^{50} \left(60x - 0.03x^3 + 0.021xy^2\right)\Big|_{-20}^{20} dy$$

$$= \frac{1}{4000}\int_{-50}^{50} \left(1200 - 240 + 0.42y^2 - (-1200 + 240 - 0.42y^2)\right) dy$$

$$= \frac{1}{4000}\int_{-50}^{50} (1920 + 0.84y^2)\, dy$$

$$= \frac{1}{4000}\left[1920y + 0.28y^3\right]_{-50}^{50}$$

$$= \frac{1}{4000}\left[1920(50) + 0.28(50)^3 - 1920(-50) - 0.28(-50)^3\right]$$

$$= \frac{1}{4000}\left[262000\right] = \boxed{65.500 \text{ ft.}}$$

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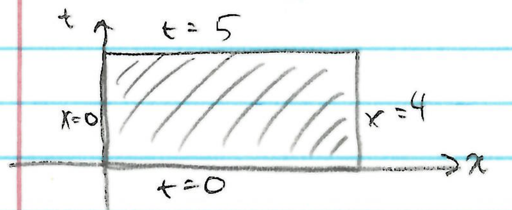
Ex 3. In a certain area, the population is $P(x,t) = \frac{8000e^{0.5t}}{1+x}$ where x is the number of miles from the center of the city, and t is the number of years after 2000. What is the average population over the first 5 years within a radius of 4 miles from the city center? (Round to whole number)



Area $= (5)(4) = 20$

$$\text{Ave} = \frac{1}{20}\int_0^4\int_0^5 \frac{8000e^{0.5t}}{1+x}\,dt\,dx$$

$$= \frac{1}{20}\int_0^4 \frac{16{,}000e^{0.5t}}{1+x}\Big|_0^5\,dx$$

$$= \frac{1}{20}\int_0^4 \frac{16{,}000e^{2.5}-16{,}000}{1+x}\,dx$$

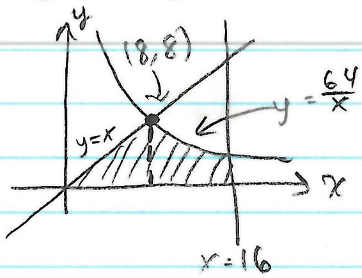
$$= \frac{16{,}000e^{2.5}-6{,}000}{20}\int_0^4 \frac{1}{1+x}\,dx$$

$$= \frac{16{,}000e^{2.5}-16{,}000}{20}\Big[\ln|1+x|\Big]_0^4$$

$$= \frac{16{,}000e^{2.5}-16{,}000}{20}\Big[\ln|5| - \ln|1|\Big]$$

$$\approx \boxed{14{,}398}$$

Ex 4. Evaluate $\iint_R 10x^2\,dA$ over the region in the first quadrant bounded by $xy = 64$ and the lines $y = x$, $y = 0$ and $x = 16$.



split into two regions
easier to do dy dx

Left Region

bounded below by $y = 0$
above by $y = x$
lowest x value: 0
highest x value: 8

Right Region

bounded below by $y = 0$
above by $y = \frac{64}{x}$
lowest x value: 8
highest x value = 16

$$\int_0^8 \int_0^x 10x^2\,dy\,dx + \int_8^{16} \int_0^{64/x} 10x^2\,dy\,dx$$

$$= \int_0^8 10x^2 y \Big|_0^x dx + \int_8^{16} 10x^2 y \Big|_0^{64/x} dx$$

$$= \int_0^8 (10x^3 - 0)\,dx + \int_8^{16} (640x - 0)\,dx$$

$$= \frac{5}{2}x^4 \Big|_0^8 + 320x^2 \Big|_8^{16}$$

$$= \frac{5}{2}(8)^4 - \frac{5}{2}(0)^4 + 320(16)^2 - 320(8)^2$$

$$= \boxed{71{,}680}$$