MA 16020

Lesson 3

Integrals Involving Logarithms

Recall that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, so $\int \frac{1}{x}\,dx = \ln|x| + C$.
Knowing the derivative of $\ln x$ can help us with substitutions. Look for a function inside another function whose derivative is in the integral.

Ex 1. $\int 3x^{-1}\cos(\ln(2x))\,dx$

Notice: $\ln(2x)$ is inside another function and its derivative $\left(\frac{2}{2x} = \frac{1}{x} = x^{-1}\right)$ is in the integral.

$u = \ln(2x)$, $\frac{du}{dx} = \frac{1}{2x}\cdot 2 = \frac{1}{x}$, so $du = \frac{1}{x}\,dx$

$$\int 3\cos(u)\,du$$
$$= 3\sin(u) + C$$
$$= \boxed{3\sin(\ln(2x)) + C}$$

Note:
We didn't integrate $\frac{1}{x}$ here, so we should not use absolute values in examples 1 + 2

Ex 2. $\int \frac{3}{x(\ln x)^5}\,dx$

$\ln x$ is inside $(\ )^5$ and $\frac{1}{x}$ is there

$u = \ln x$, $du = \frac{1}{x}\,dx$

$$3\int u^{-5}\,du = 3\left(-\tfrac{1}{4}u^{-4}\right) + C$$
$$= \boxed{-\frac{3}{4}(\ln x)^{-4} + C}$$
$$= \boxed{-\frac{3}{4(\ln x)^4} + C}$$

Sometimes you will perform a substitution which will lead you to an integral like $\int \frac{1}{u}\,du$, which has a value of $\ln|u| + C$.

Rember that if you have a fraction like $\frac{f(x)}{g(x)}$, $g(x)$ is the "inside function" since $\frac{f(x)}{g(x)} = f(x)\cdot\frac{1}{g(x)}$ and $g(x)$ is "inside" $\frac{1}{x}$. But always look for the derivative to be present.

Ex 3. $\int \tan(2x)\,dx$

Notice: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\int \frac{\sin(2x)}{\cos(2x)}\,dx$

$u = \cos(2x)$, $\frac{du}{dx} = -2\sin(2x)$, $-\frac{1}{2}du = \sin(2x)\,dx$

$-\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| + C$

$= \boxed{-\frac{1}{2}\ln|\cos(2x)| + C}$

Are absolute values necessary here? Yes! $\cos(2x)$ can be negative!

Ex 4. $\int \frac{2t}{5+3t^2}\,dt$

Try $u = 5+3t^2$, $du = 6t\,dt$, so $\frac{1}{6}du = t\,dt$

$\frac{2}{6}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|5+3t^2| + C$

Notice: $5+3t^2$ is always positive, so $\boxed{\frac{1}{3}\ln(5+3t^2) + C}$

Do not use absolute values since they are unnecessary

Ex 5. $\int \frac{1}{3x^{5/6}(1+x^{1/6})}\,dx$

$u = 1+x^{1/6}$, $du = \frac{1}{6}x^{-5/6}dx$, $6\,du = \frac{1}{x^{5/6}}\,dx$

$\frac{1}{3}\int \frac{1}{(1+x^{1/6})}\frac{1}{x^{5/6}}\,dx = \frac{6}{3}\int \frac{1}{u}\,du$

$= 2\ln|u| + C$

$= 2\ln|1+x^{1/6}| + C = \boxed{2\ln(1+x^{1/6}) + C}$

$x^{1/6} = \sqrt[6]{x}$ is an <u>even</u> root, and <u>even</u> roots are are never negative.
<u>odd</u> roots can be negative, so keep absolute values for <u>odd</u> roots

Reminders about application problems:

- actual change in $f(x)$ from a to b is $\int_a^b f'(x)\,dx$

  ↳ To find the value of $f(x)$ after b time units given $f'(x)$ and an initial condition, you can find $f(x) = \int f'(x)\,dx + C$, use initial condition to find C and plug in b, <u>or</u> "initial condition" $+ \int_0^b f'(x)\,dx$.

- The derivative is the slope of the tangent line
- average value of $f(x)$ from a to b is $\frac{1}{b-a}\int_a^b f(x)\,dx$ (using the actual function <u>itself</u>, not its derivative)

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Integration has Techniques, not Rules

- $\int \frac{1}{1+x}\,dx = \ln|1+x| + C$ (let $u = 1+x$)
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$ (knowing $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$)
- $\int \frac{x}{1+x}\,dx = 1+x - \ln|1+x| + C$ (let $u = 1+x$, also $x = u-1$)
- $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| + C$ (let $u = 1+x^2$)
- $\int \frac{1}{(1+x)^2}\,dx = -\frac{1}{1+x} + C$ (let $u = 1+x$)
- $\int \frac{x}{(1+x)^2}\,dx = \ln|1+x| + \frac{1}{1+x} + C$ (let $u = 1+x$, also $x = u-1$)

All these integrals look pretty similar, but the results can be quite different and can use different methods to find.
You actually need to think carefully to do integrals.