

MA 16020

Lesson 30

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## Systems of Equations and Matrices

Systems of linear equations can look like

$$\begin{cases} x + 3y = 5 \\ 2x - y = 7 \end{cases} \quad \text{or} \quad \begin{cases} x - 5y + z = 4 \\ 3x + 2z = 5 \\ 7x - y - 4z = 8 \end{cases}$$

(or using more unknowns and more equations)

Linear systems can be solved using substitution or using elimination (or both). Note how, in elimination, you can multiply a single equation by a nonzero constant and you can add equations together. And you do this to eliminate a variable.

We can write systems of equations as augmented matrices, to help condense information.

Ex 1. Write the system as an augmented matrix.

$$\begin{cases} x - 5y + z = 4 \\ 3x + 2z = 5 \\ 7x - y - 4z = 8 \end{cases} \quad \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[ \begin{array}{ccc|c} x & y & z & \\ 1 & -5 & 1 & 4 \\ 3 & 0 & 2 & 5 \\ 7 & -1 & -4 & 8 \end{array} \right]$$

Based on the techniques of elimination, you can perform Elementary Row Operations on an augmented matrix, and it won't change the solutions of the system.

1. Swap two rows (e.g.,  $R_1 \leftrightarrow R_2$ )
2. Multiply a row by a nonzero number (e.g.,  $\frac{1}{2}R_2 \rightarrow R_2$ ,  $\frac{1}{2}R_2$  replacing  $R_2$ )
3. Adding a multiple of one row to another row (e.g.,  $2R_1 + R_2 \rightarrow R_2$ , replacing  $R_2$  with 2 times  $R_1$  added to  $R_2$ )

## Gaussian Elimination

Given a system of equations,

1. Write the system as an augmented matrix
2. Perform elementary row operations to get the  $(1,1)$ -entry to be a 1 (the  $(i,j)$ -entry of a matrix is the entry in row  $i$  and column  $j$ )
3. Perform elementary row operations to get all entries below  $(1,1)$  to be 0 (use the 1 in  $(1,1)$ )
4. Perform elementary row operations to get the  $(2,2)$ -entry to be a 1.
5. Perform elementary row operations to get all entries below  $(2,2)$  to be 0 (use the 1 in  $(2,2)$ )  
etc.

6. The process ends when your matrix is in

Row Echelon Form  $\left[ \begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array} \right]$  or  $\left[ \begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array} \right]$

(1's along diagonal and 0's under the diagonal)

7. Now use substitution to solve your new system.

Ex 2. A goldsmith has two alloys of gold, the first having a purity of 70% and the second having a purity of 82%. If  $x$  grams of the first alloy are mixed with  $y$  grams of the second, obtaining 100 grams of alloy with 78% gold, find  $x$  to the nearest gram.

grams:  $x + y = 100$

percentage gold:  $0.7x + 0.82y = 0.78 \cdot 100$

$$0.7x + 0.82y = 78$$

$$\begin{cases} 0.7x + 0.82y = 78 \\ x + y = 100 \end{cases}$$

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$$\begin{bmatrix} 0.7 & 0.82 & | & 78 \\ 1 & 1 & | & 100 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & | & 100 \\ 0.7 & 0.82 & | & 78 \end{bmatrix}$$

$$\xrightarrow{-0.7R_1 + R_2} \begin{bmatrix} 1 & 1 & | & 100 \\ 0 & 0.12 & | & 8 \end{bmatrix} \xrightarrow{\frac{1}{0.12}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 1 & | & 100 \\ 0 & 1 & | & \frac{200}{3} \end{bmatrix} \text{ REF}$$

$$\begin{cases} x + y = 100 \\ y = \frac{200}{3} \end{cases} \Rightarrow x + \frac{200}{3} = 100 \Rightarrow x = 100 - \frac{200}{3} \approx \boxed{33 \text{ grams}}$$

Ex 3. An object is moving vertically where  $a$  is constant acceleration,  $v$  is velocity at  $t=0$  and  $h$  is height at  $t=0$  is represented by the equation  $s(t) = \frac{1}{2}at^2 + vt + h$ .  
When  $t=1$ ,  $s=38$ ; when  $t=2$ ,  $s=50$ ; when  $t=3$ ,  $s=38$ . Find the equation for  $s(t)$ .

$$s(1) = \frac{1}{2}a + v + h = 38, \quad s(2) = \frac{1}{2}a(2)^2 + v(2) + h = 2a + 2v + h = 50,$$

$$s(3) = \frac{1}{2}a(3)^2 + v(3) + h = \frac{9}{2}a + 3v + h = 38$$

$$\begin{array}{ccc} a & v & h \\ \begin{bmatrix} \frac{1}{2} & 1 & 1 & | & 38 \\ 2 & 2 & 1 & | & 50 \\ \frac{9}{2} & 3 & 1 & | & 38 \end{bmatrix} & \xrightarrow{2R_1 \rightarrow R_1} & \begin{bmatrix} 1 & 2 & 2 & | & 76 \\ 2 & 2 & 1 & | & 50 \\ \frac{9}{2} & 3 & 1 & | & 38 \end{bmatrix} \end{array}$$

$$\xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 2 & 2 & | & 76 \\ 0 & -2 & -3 & | & -102 \\ \frac{9}{2} & 3 & 1 & | & 38 \end{bmatrix} \xrightarrow{-\frac{9}{2}R_1 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & 2 & 2 & | & 76 \\ 0 & -2 & -3 & | & -102 \\ 0 & -6 & -8 & | & -304 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & 2 & | & 76 \\ 0 & 1 & \frac{3}{2} & | & 51 \\ 0 & -6 & -8 & | & -304 \end{bmatrix} \xrightarrow{6R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & 2 & 2 & | & 76 \\ 0 & 1 & \frac{3}{2} & | & 51 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \text{ REF}$$

$$\begin{cases} a + 2v + 2h = 76 \\ v + \frac{3}{2}h = 51 \Rightarrow v + \frac{3}{2}(2) = 51 \Rightarrow v = 48 \\ h = 2 \quad a + 2(48) + 2(2) = 76 \Rightarrow a = -24 \end{cases}$$

$$s(t) = \frac{1}{2}(-24)t^2 + 48t + 2$$

$$\boxed{s(t) = -12t^2 + 48t + 2}$$

A system is called inconsistent if it has no solutions (e.g., if you get  $\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{array} \right]$ )

It is called consistent independent if it has one solution (e.g., examples 2 + 3)

It is called consistent dependent if it has infinitely many solutions, then the solutions depend on the value of one of the variables,

e.g., if you end up with

$$\left[ \begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ then you have } \begin{cases} x + 2y + 2z = 4 \\ y + 3z = 8 \\ 0 = 0 \end{cases}$$

Usually, we set  $z = t$ .

$$\text{then } y + 3t = 8 \Rightarrow y = 8 - 3t$$

$$\text{then } x + 2(8 - 3t) + 2t = 4$$

$$\Rightarrow x + 16 - 6t + 2t = 4$$

$$\Rightarrow x = 4t - 12$$

So solutions are of the form

$$\underline{(4t - 12, 8 - 3t, t)}$$

Any value plugged in to  $t$  is a solution!