MA 16020
Lesson 32
Matrix Operations

An $m \times n$ matrix is an array of numbers having m rows and n columns. The (i,j)-entry of a matrix is the number in row i and column j.

$$\begin{array}{c} \\ \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix} \end{array}$$

The $(2,3)$-entry is 4.

Scalar multiplication: Any matrix can be multiplied by a scalar (number) by multiplying all entries of the matrix by that scalar.

Matrix Addition: If A and B are both $m \times n$ matrices (same # of rows + same # of columns) then $A+B$ is found by adding corresponding entries together.

Matrix Subtraction: If A and B are both $m \times n$ matrices, $A - B := A + (-1)B$

Ex 1. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$,

find $2A + 3B$

$$2A = 2\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix}, \quad 3B = 3\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 12 & -3 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 10 & 3 \end{bmatrix}$$

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Ex 2. If $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$,

find $3A - B$.

$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ -3 & 6 & 3 \\ 9 & 0 & 0 \end{bmatrix}$, $(-1)B = -1\begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -3 \\ 0 & -2 & -1 \\ -1 & 1 & -2 \end{bmatrix}$

$3A - B = 3A + (-1)B = \begin{bmatrix} 3 & 0 & 3 \\ -3 & 6 & 3 \\ 9 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -3 & -3 \\ 0 & -2 & -1 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 4 & 2 \\ 8 & 1 & -2 \end{bmatrix}$

Vectors and Vector Multiplication

An $m \times 1$ matrix is called a column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$
and a $1 \times n$ matrix is called a row vector $[a_1 \cdots a_n]$

a $1 \times n$ row vector can be multiplied by an $n \times 1$ column vector by multiplying corresponding components and adding them together

e.g., $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 2\cdot3 + 0\cdot(-1) + 1\cdot2$
$= 6 + 0 + 2 = 8$

$\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1\cdot(-1) + 3\cdot2 = -1+6 = 5$

Notice: An $m \times n$ matrix $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$
is made up of m row vectors of size n and n column vectors of size m

Matrix multiplication is based off vector multiplication.

<u>Matrix Multiplication</u> If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then AB is defined and is an $m \times k$ matrix

$$\left(m \times n \text{ times } n \times k \right)$$

match

size of product

The (i,j)-entry of AB is obtained by multiplying the ith row vector of A by the j^{th} row vector of B.

<u>Ex 3.</u> If $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$

find the $(1,2)$-entry of AB

multiply row 1 of A: $\begin{bmatrix} 1 & 2 \end{bmatrix}$ by

column 2 of B $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 1 \cdot (-1) + 2 \cdot 3 = -1 + 6 = \boxed{5}$$

<u>Ex 4.</u> If $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 4 & -1 \end{bmatrix}$,

find AB and BA

$A = 2 \times \underbrace{3 \;,\; B = 3}_{\checkmark} \times 2$, AB is 2×2

$$B \rightarrow \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$\overset{A}{\downarrow}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1+0-8 & 1+0+2 \\ 0-2+4 & 0+3-1 \end{bmatrix}}_{AB} = \begin{bmatrix} -7 & 3 \\ 2 & 2 \end{bmatrix}$$

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$B = 3\times \underbrace{2 \quad A = 2}\times 3$, BA is 3×3

$$\begin{array}{cc} & A \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \\ B \downarrow \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 4 & -1 \end{bmatrix} & \begin{bmatrix} 1+0 & 0+1 & -2+1 \\ -2+0 & 0+3 & 4+3 \\ 4+0 & 0-1 & -8-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & 7 \\ 4 & -1 & -9 \end{bmatrix} \end{array}$$

Notice $AB \neq BA$, so matrix multiplication is <u>not</u> commutative.

<u>Ex 5</u>. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, find A^2.

$A^2 = AA$, which is okay since A is 2×2

$$\begin{array}{cc} & A \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \\ A \downarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 1+0 & -1-3 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 9 \end{bmatrix} \end{array}$$