MA 16020
Lesson 33
Inverse Matrices

In this lesson, we work with square matrices $(n \times n)$.

The $n \times n$ identity matrix I_n is the $n \times n$ matrix with 1's on the diagonal and 0's everywhere else. e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For any $n \times n$ matrix A, $AI_n = I_nA = A$.

When talking about identities, we often also talk about inverses. If A is an $n \times n$ matrix, we want an $n \times n$ matrix A^{-1} so that $AA^{-1} = A^{-1}A = I_n$.

Not every $n \times n$ matrix has an inverse, but if it does, you can find it with the following method:
Take the augmented matrix $[A \mid I_n]$
Get this into RREF (use Gauss-Jordan Elimination)
and you will have $[I_n \mid A^{-1}]$

Ex 1. Find A^{-1} where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{-3R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \text{ so } A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ ✓

$A^{-1}A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

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Ex 2. Find A^{-1} where $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \quad \text{so } A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$A^{-1}A = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ ✓

Solving systems with an inverse matrix

Given a system, let A be the coefficient matrix of the system (the left part of the augmented matrix) and let $\vec{x}$ be the variable column vector and $\vec{b}$ be the column vector of values.

The $A\vec{x} = \vec{b}$

(e.g., $\begin{matrix} 3x + 4y = 4 \\ x - 2y = 3 \end{matrix}$, $A = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

So $\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$)

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If A^{-1} exists, then you can multiply to get

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I_n\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

So you can find the solutions by multiplying the value column vector on the left by the inverse of the coefficient matrix.

Ex 3. Solve the system

$$\begin{cases} x - 3y = -8 \\ 2x + y = 5 \end{cases}$$

$A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 7 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} \end{array}\right]$$

$$\xrightarrow{3R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} \end{array}\right] \quad \text{so } A^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$A^{-1}\vec{b} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -8 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{8}{7} + \frac{15}{7} \\ \frac{16}{7} + \frac{5}{7} \end{bmatrix} = \begin{bmatrix} \frac{7}{7} \\ \frac{21}{7} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

so $x = 1$ and $y = 3$

$$\boxed{(1, 3)}$$

<u>Ex 4</u>. Solve $\begin{cases} x - y + z = 2 \\ 2x + 2y + 2z = 8 \\ x - y - z = -2 \end{cases}$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 8 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -2 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \quad \text{so } A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1}\,\vec{b} \qquad \begin{bmatrix} 2 \\ 8 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0+2-1 \\ -1+2+0 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$x = 1, \ y = 1, \ z = 2$

$$\boxed{(1, 1, 2)}$$