MA 16020
Lesson 34
Determinants

If you have $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and try to find the inverse, you will end up with $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This is a valid matrix whenever $ad - bc \neq 0$.

Determinants of a 2×2 Matrix

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\underline{\det(A)} = \underline{|A|} = ad - bc$ is called the <u>determinant</u> of A.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

A matrix is called <u>nonsingular</u> if it is invertible and is called <u>singular</u> if it is not invertible.

<u>Theorem</u>. A square matrix A is nonsingular if and only if $\det A \neq 0$. Equivalently, it is singular if and only if $\det A = 0$.

<u>Ex 1.</u> Determine if A, B are singular or nonsingular.

$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

$\det(A) = (2)(2) - (-4)(-1) = 4 - 4 = 0$

$\det(B) = (1)(-1) - (3)(2) = -1 - 6 = -7$

$\det A = 0$, so A is singular (i.e., not invertible, i.e., A^{-1} does not exist)

$\det B \neq 0$, so B is nonsingular (i.e., invertible, i.e., B^{-1} exists)

Fun example: Given a 2-variable function $f(x,y)$, the Hessian matrix of $f(x,y)$ is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}. \quad \det H = f_{xx}f_{yy} - (f_{xy})^2 \quad (\text{since } f_{xy} = f_{yx})$$

(the function $D(x,y)$ in the second derivative test)

<u>Ex 2</u>. Find the values of x which make the statement true $\begin{vmatrix} x+2 & 1 \\ 3 & x \end{vmatrix} = 0$

$$(x+2)(x) - (3)(1) = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$\boxed{x = -3 \text{ or } x = 1}$$

<u>Minors and Cofactors</u>

Determinants for 3×3 and larger matrices a trickier, so we use minors and cofactors.

The <u>(i,j)-minor M_{ij}</u> is the determinant of the matrix resulting from deleting row i and column j from the current matrix.

e.g., $\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{2} \\ 3 & 4 & \cancel{-1} \\ 1 & 1 & \cancel{2} \end{bmatrix}$, $\quad M_{13} = \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = (3)(1) - (1)(4) = -1$

The <u>(i,j)-cofactor C_{ij}</u> is $(-1)^{i+j} M_{ij}$

So in the above example, $C_{13} = (-1)^{1+3}(-1)$
$$= (-1)^4(-1)$$
$$= -1$$

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Determinants of 3x3 (and larger matrices)

You can take a cofactor expansion along any row or any column of the matrix.

Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the cofactor expansion along row 3, for example, is

$a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

The cofactor expansion along column 2 is

$a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

Any cofactor expansion along any row or any column gives the determinant
(rows/columns with 0's and 1's tend to be easier)

Ex 3. Find det A, $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 1 \\ 4 & -2 & 3 \end{bmatrix}$

Row 2 looks nicest, so expand along row 2:

$\det(A) = 0 \cdot C_{21} + 2 \cdot C_{22} + 1 \cdot C_{23}$
$= 2C_{22} + C_{23}$

M_{22}: $\begin{bmatrix} -1 & \cancel{3} & 2 \\ \cancel{0} & \cancel{2} & \cancel{1} \\ 4 & \cancel{-2} & 3 \end{bmatrix}$, $M_{22} = \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} = (-1)(3) - (4)(2) = -11$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4(-11) = -11$

M_{23}: $\begin{bmatrix} -1 & 3 & \cancel{2} \\ \cancel{0} & \cancel{2} & \cancel{1} \\ 4 & -2 & \cancel{3} \end{bmatrix}$, $M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} = (-1)(-2) - (4)(3) = -10$

$C_{23} = (-1)^{2+3}(-10) = (-1)^5(-10) = 10$

$\det(A) = 2C_{22} + C_{23} = 2(-11) + 10 = \boxed{-12}$

Ex 4. Find det A, $A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 3 & -2 \\ 4 & 2 & 1 \end{bmatrix}$

Column 3 looks nicest, so expand along column 3:

$1 \cdot C_{13} + (-2) \cdot C_{23} + 1 \cdot C_{33}$

$\det(A) = C_{13} - 2C_{23} + C_{33}$

$M_{13}: \begin{bmatrix} \cancel{3} & \cancel{2} & \cancel{1} \\ -1 & 3 & \cancel{-2} \\ 4 & 2 & \cancel{1} \end{bmatrix}$, $M_{13} = \begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} = (-1)(2) - (4)(3) = -14$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4(-14) = -14$

$M_{23}: \begin{bmatrix} 3 & 2 & \cancel{1} \\ \cancel{-1} & \cancel{3} & \cancel{-2} \\ 4 & 2 & \cancel{1} \end{bmatrix}$, $M_{23} = \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = (3)(2) - (4)(2) = -2$

$C_{23} = (-1)^{2+3} M_{23} = (-1)^5(-2) = 2$

$M_{33}: \begin{bmatrix} 3 & 2 & \cancel{1} \\ -1 & 3 & \cancel{-2} \\ \cancel{4} & \cancel{2} & \cancel{1} \end{bmatrix}$, $M_{33} = \begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} = (3)(3) - (-1)(2) = 11$

$C_{33} = (-1)^{3+3} M_{33} = (-1)^6(11) = 11$

$\det A = C_{13} - 2C_{23} + C_{33}$
$= -14 - 2(2) + 11$
$= \boxed{-7}$