MA 16020
Lesson 35
Eigenvalues and eigenvectors (Part I)

Matrices can be thought of as transformations of space: A point in space (x,y) can be represented by the vector $\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then sends the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

Often, we care about vectors which only change length but not direction by such a transformation of space: i.e., vectors $\vec{x}$ so that $A\vec{x} = \lambda \vec{x}$ for some scalar λ.

Provided $\vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, if $A\vec{x} = \lambda\vec{x}$, we say that λ is an <u>eigenvalue</u> of A and $\vec{x}$ is an <u>eigenvector</u> of A associated to λ

<u>Ex 1</u>. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$. Which of the following are eigenvectors of A?

$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$A\vec{a}$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ <u>is</u> an eigenvector of A associated to 1

$A\vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

for any λ, so $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is <u>not</u> an eigenvector of A

$A\vec{c}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A associated to -1

$A\vec{d}$ $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ for any λ, so $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is <u>not</u> an eigenvector of A

<u>How to find eigenvalues and eigenvectors</u>

We know $A\vec{x} = \lambda\vec{x}$. Subtracting $A\vec{x}$, we get
$\vec{0} = \lambda\vec{x} - A\vec{x}$. Factoring out $\vec{x}$, we get
$\vec{0} = (\lambda I_n - A)\vec{x}$

Since we do not allow $\vec{x} = \vec{0}$, this is only possible if $\lambda I_n - A$ is singular, i.e., $\det(\lambda I_n - A) = 0$.

To find eigenvalues of an $n \times n$ matrix A, compute $\det(\lambda I_n - A)$. You will get a polynomial in λ. Find all roots of this polynomial. These roots are the eigenvalues.

Given a specific eigenvalue λ_1, all eigenvectors $\vec{x}_1$ associated to λ_1 satisfy $(\lambda_1 I_n - A)\vec{x}_1 = \vec{0}$, so take the augmented matrix $[\lambda_1 I_n - A \mid \vec{0}]$ and solve the system. There will be infinitely many solutions. All such vectors are eigenvectors associated to λ_1.

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Ex 2. Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Find $\det\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right) = \begin{vmatrix} \lambda-1 & 0 \\ -2 & \lambda+1 \end{vmatrix}$

$= (\lambda-1)(\lambda+1) - 0 = \lambda^2 - 1$

$\lambda^2 - 1 = 0$ when $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\lambda_1 = -1$, $\lambda_2 = 1$ are the eigenvalues

$\lambda_1 = -1$: $\left[\begin{array}{cc|c} -1-1 & 0 & 0 \\ -2 & -1+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 0 & 0 \\ -2 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ -2 & 0 & 0 \end{array}\right]$

$\xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x = 0 \\ 0 = 0 \end{cases}$ y can be anything, say $y = t$.

$\vec{x}_1 = \begin{bmatrix} 0 \\ t \end{bmatrix}$ can choose $t = 1$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$: $\left[\begin{array}{cc|c} 1-1 & 0 & 0 \\ -2 & 1+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x - y = 0 \\ 0 = 0 \end{cases} \Rightarrow x = y$

y can be anything, say $y = t$

$\vec{x}_2 = \begin{bmatrix} t \\ t \end{bmatrix}$, can choose $t = 1$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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<u>Ex 3.</u> Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right) = \begin{vmatrix} \lambda-2 & 4 \\ 1 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+1) - 4$$

$$= \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$$

roots: $\lambda_1 = -2$, $\lambda_2 = 3$ $\leftarrow$ eigenvalues

$\lambda_1 = -2$: $\left[\begin{array}{cc|c} -2-2 & 4 & 0 \\ 1 & -2+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & 4 & 0 \end{array}\right]$

$\xrightarrow{4R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x - y = 0 \Rightarrow x = y$ let $y = t$

$\vec{x}_1 = \begin{bmatrix} t \\ t \end{bmatrix}$, can let $t = 1$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$: $\left[\begin{array}{cc|c} 3-2 & 4 & 0 \\ 1 & 3+1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 1 & 4 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x + 4y = 0 \Rightarrow x = -4y$ let $y = t$

$\vec{x}_2 = \begin{bmatrix} -4t \\ t \end{bmatrix}$, can let $t = 1$: $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$