MA 16020

Lesson 36

Eigenvalues and eigenvectors (Part 3)

Ex 1. $A = \begin{bmatrix} -5 & 2 & 1 \\ -2 & -1 & 1 \\ -4 & 2 & 0 \end{bmatrix}$ has -3 as an eigenvalue.

Is $\begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$ an eigenvector of A associated to -3?

$$\begin{bmatrix} -5 & 2 & 1 \\ -2 & -1 & 1 \\ -4 & 2 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 20-4-4 \\ 8+2-4 \\ 16-4+0 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 12 \end{bmatrix} \qquad -3\begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 12 \end{bmatrix}$$

yes

Ex 2. Find the eigenvalues and associated eigenvectors of $A = \begin{bmatrix} -29 & 14 & 20 \\ -31 & 16 & 20 \\ -17 & 7 & 15 \end{bmatrix}$

$$\begin{vmatrix} \lambda+29 & -14 & -20 \\ 31 & \lambda-16 & -20 \\ 17 & -7 & \lambda-15 \end{vmatrix} = (\lambda+29)C_{11} + (-14)C_{12} + (-20)C_{13}$$

$$C_{11} = (-1)^2 \begin{vmatrix} \lambda-16 & -20 \\ -7 & \lambda-15 \end{vmatrix} = \lambda^2 - 31\lambda + 100$$

$$C_{12} = (-1)^3 \begin{vmatrix} 31 & -20 \\ 17 & \lambda-15 \end{vmatrix} = -(31\lambda - 125) = -31\lambda + 125$$

$$C_{13} = (-1)^4 \begin{vmatrix} 31 & \lambda-16 \\ 17 & -7 \end{vmatrix} = -217 - 17\lambda + 272 = -17\lambda + 55$$

$$(\lambda+29)(\lambda^2 - 31\lambda + 100) - 14(-31\lambda + 125) - 20(-17\lambda + 55)$$

$$= \lambda^3 \underline{- 31\lambda^2} + 100\lambda + \underline{29\lambda^2} - 899\lambda + 2900 + 434\lambda - 1750 + 340\lambda - 1100$$

$$= \lambda^3 - 2\lambda^2 - 25\lambda + 50$$

$$= \lambda^2(\lambda - 2) - 25(\lambda - 2)$$

$$= (\lambda - 2)(\lambda^2 - 25)$$

$$= (\lambda - 2)(\lambda + 5)(\lambda - 5)$$

$$\lambda_1 = -5, \quad \lambda_2 = 2, \quad \lambda_3 = 5$$

MA 16020

Lesson 36

$\lambda_1 = -5: \left[\begin{array}{ccc|c} -5+29 & -14 & -20 & 0 \\ 31 & -5-16 & -20 & 0 \\ 17 & -7 & -5-15 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 24 & -14 & -20 & 0 \\ 31 & -21 & -20 & 0 \\ 17 & -7 & -20 & 0 \end{array}\right]$

$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x-2z=0 \\ y-2z=0 \end{array} \Rightarrow \begin{array}{l} x=2z \\ y=2z \end{array}$

let $z=t$

$\vec{x}_1 = \begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix}$ can choose $t=1$: $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 2: \left[\begin{array}{ccc|c} 31 & -14 & -20 & 0 \\ 31 & -14 & -20 & 0 \\ 17 & -7 & -13 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{array}{l} x-2z=0 \\ y-3z=0 \end{array} \Rightarrow \begin{array}{l} x=2z \\ y=3z \end{array}$ let $z=t$

$\vec{x}_2 = \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix}$ can choose $t=1$: $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_3 = 5: \left[\begin{array}{ccc|c} 34 & -14 & -20 & 0 \\ 31 & -11 & -20 & 0 \\ 17 & -7 & -10 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{array}{l} x-z=0 \\ y-z=0 \end{array} \Rightarrow \begin{array}{l} x=z \\ y=z \end{array}$ let $z=t$

$\vec{x}_3 = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$ can choose $t=1$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

<u>Rational Root Theorem</u>. Given a polynomial with 1 as the leading coefficient, all of the <u>rational</u> roots are factors of the constant term.

MA 16020
Lesson 36

Ex 3. If $\det(\lambda I_3 - A) = \lambda^3 - 13\lambda + 12$, find the eigenvalues of A.

leading coefficient is 1, 12 is constant term.
factors of 12 : $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
Plug into polynomial until you find a root
$(1)^3 - 13(1) + 12 = 0$ so 1 is a root

polynomial long division:

$$\begin{array}{r l}
 & \lambda^2 + \lambda - 12 \\
\lambda - 1 \,\big) & \lambda^3 - 13\lambda + 12 \\
 & -(\lambda^3 - \lambda^2) \\
\hline
 & \lambda^2 - 13\lambda + 12 \\
 & -(\lambda^2 - \lambda) \\
\hline
 & -12\lambda + 12 \\
 & -(-12\lambda + 12) \\
\hline
 & 0
\end{array}$$

$(\lambda - 1)(\lambda^2 + \lambda - 12)$
$= (\lambda - 1)(\lambda - 3)(\lambda + 4)$

$$\boxed{\lambda_1 = -4, \ \lambda_2 = 1, \ \lambda_3 = 3}$$