MA 16020
Lesson 4
Integration by Parts (Part 1)

Many of you are tempted to integrate a product by integrating its factors, but $\int u(x)\cdot v(x)\,dx$ is **not** $\int u(x)\,dx \cdot \int v(x)\,dx$.

Recall: The product rule states that
$$(uv)' = uv' + vu', \text{ or } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
Multiplying both sides by dx gives
$$d(uv) = u\,dv + v\,du$$
Integrating both sides, we get
$$\int d(uv) = \int u\,dv + \int v\,du$$
Subtracting $\int v\,du$ from both sides, we get
$$\int u\,dv = \int d(uv) - \int v\,du$$
$$\boxed{\int u\,dv = uv - \int v\,du}$$
Memorize this formula!

The Technique of Integration by Parts

If you are trying to integrate a product of functions, and u-substitution does not work,

1. Choose the function which is easiest to differentiate, and set it equal to u.
2. Take the remaining function and dx, and set it equal to dv.
3. Differentiate u to get du.
4. Integrate dv to get v.
5. Plug into formula $\int u\,dv = uv - \int v\,du$.
6. Finish the integral.

Oftentimes, the best function to choose for u follows

L A T E
Logs
Algebra
Trig
Exponential

Ex 1. $\int_1^3 x(x-2)^6\,dx$

(Can do u-sub! But let's get practice with integration by parts.)

It's easy to differentiate x, so let $u = x$

Then let $dv = (x-2)^6 dx$

$\frac{du}{dx} = 1$, so $du = dx$

$\int dv = \int (x-2)^6 dx = \frac{1}{7}(x-2)^7$, so $v = \frac{1}{7}(x-2)^7$

$\int u\,dv = uv - \int v\,du$

$$\int_1^3 x(x-2)^6\,dx = \left(x \cdot \frac{1}{7}(x-2)^7 - \int \frac{1}{7}(x-2)^7\,dx\right)\Big|_1^3$$

$$= \left(\frac{x(x-2)^7}{7} - \frac{1}{56}(x-2)^8\right)\Big|_1^3$$

$$= \left(\frac{(3)(1)^7}{7} - \frac{(1)^8}{56}\right) - \left(\frac{(1)(-1)^7}{7} - \frac{(-1)^8}{56}\right)$$

$$= \frac{3}{7} - \frac{1}{56} + \frac{1}{7} + \frac{1}{56} = \boxed{\frac{4}{7}}$$

Ex 2. $\int (t-1)e^{1-t}\,dt$ (u-sub does not work here)

By LATE, $\underline{u = t-1}$

so $\underline{dv = e^{1-t}dt}$

$\frac{du}{dt} = 1$, so $\underline{du = dt}$

$v = \int dv = \int e^{1-t}dt$ ($w = 1-t$, $dw = -dt$, $-dw = dt$)

$-\int e^w dw = -e^w = -e^{1-t}$, so $\underline{v = -e^{1-t}}$

$\int u\,dv = uv - \int v\,du$

$$\int (t-1)e^{1-t}dt = (t-1)(-e^{1-t}) - \int -e^{1-t}\,dt$$

$$= \boxed{-(t-1)e^{1-t} - e^{1-t} + C}$$

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Ex 3. $\int 5x\cos(3x)\,dx$ (u-sub does not work here)

By LATE, $u = 5x$

so $dv = \cos(3x)\,dx$

$\frac{du}{dx} = 5$, so $du = 5\,dx$

$v = \int dv = \int \cos(3x)\,dx$ ($w = 3x$, $dw = 3\,dx$, $\frac{1}{3}dw = dx$)

$\frac{1}{3}\int \cos(w)\,dw = \frac{1}{3}\sin(w) = \frac{1}{3}\sin(3x)$; $v = \frac{1}{3}\sin(3x)$

$\int u\,dv = uv - \int v\,du$

$\int 5x\cos(3x) = 5x \cdot \frac{1}{3}\sin(3x) - \int \frac{1}{3}\sin(3x) \cdot 5\,dx$

$= \frac{5}{3}x\sin(3x) - \frac{5}{3}\int \sin(3x)\,dx$

$w = 3x$, $dw = 3\,dx$, $\frac{1}{3}dw = dx$

$\int \sin(3x)\,dx = \frac{1}{3}\int \sin(w)\,dw = -\frac{1}{3}\cos(3x)$

$\frac{5}{3}x\sin(3x) - \frac{5}{3}\left(-\frac{1}{3}\cos(3x)\right) + C$

$\boxed{\frac{5}{3}x\sin(3x) + \frac{5}{9}\cos(3x) + C}$

Ex 4. $\int 7x\ln(x^8)\,dx$

By LATE, $u = \ln(x^8)$

so $dv = 7x\,dx$

$\frac{du}{dx} = \frac{1}{x^8} \cdot 8x^7 = \frac{8}{x}$, so $du = \frac{8}{x}dx$

$v = \int dv = \int 7x\,dx = \frac{7}{2}x^2$, so $v = \frac{7}{2}x^2$

$\int u\,dv = uv - \int v\,du$

$\int 7x\ln(x^8)\,dx = \ln(x^8) \cdot \frac{7}{2}x^2 - \int \frac{7}{2}x^2 \cdot \frac{8}{x}\,dx$

$= \frac{7}{2}x^2\ln(x^8) - 28\int x\,dx$

$= \frac{7}{2}x^2\ln(x^8) - 28 \cdot \frac{1}{2}x^2 + C$

$= \boxed{\frac{7}{2}x^2\ln(x^8) - 14x^2 + C}$

Ex 5. $\int \frac{3x^3}{\sqrt{7+x^2}}\,dx$ (can do u-sub $u=7+x^2$ and $x^2=u-7$)

$$\int 3x^2 \cdot x\,(7+x^2)^{-1/2}\,dx$$

$u=3x^2$ $\qquad$ $dv = x(7+x^2)^{-1/2}\,dx$

$du = 6x\,dx$ $\qquad$ $w=7+x^2$, $dw=2x\,dx$, $\frac{1}{2}dw = x\,dx$

$\frac{1}{2}\int w^{-1/2}\,dw = w^{1/2} = (7+x^2)^{1/2}$

$v = (7+x^2)^{1/2}$

$$\int \frac{3x^3}{\sqrt{7+x^2}}\,dx = 3x^2\cdot(7+x^2)^{1/2} - \int (7+x^2)^{1/2}\cdot 6x\,dx$$

$$= 3x^2\sqrt{7+x^2} - 6\int x(7+x^2)^{1/2}\,dx$$

$u=7+x^2$

$du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$

$$\int x(7+x^2)^{1/2}dx = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{1}{3}(7+x^2)^{3/2}$$

$$3x^2\sqrt{7+x^2} - 6\left(\frac{1}{3}(7+x^2)^{3/2}\right) + C$$

$$\boxed{3x^2\sqrt{7+x^2} - 2(7+x^2)^{3/2} + C}$$

Sometimes you may need to perform integration by parts twice (doing it again on $\int v\,du$, for example)

$\int (6z^2+5)e^z\,dz$ is an example.