MA 16020
Lesson 6
Intro to Differential Equations

A differential equation (diff eq) is an equation relating a function and its derivative(s).

For example, $y = y'$ is a diff eq. A solution to a diff eq is a function y which satisfies the equation. In the example $y = y'$, $y = Ce^x$ is a solution for any constant C. A solution with a "C" is a general solution

A diff eq together with an initial condition is an initial value problem (IVP). The initial condition determines the value of C. The solution to an IVP does not have a "C" and is a particular solution.

In the next 3 lessons, we study separable diff eqs. In such diff eqs, we can get all y's on one side and all x's or t's on the other, then integrate.

Ex 1. Solve the diff eq $y' = e^{y-2x}$

$$\frac{dy}{dx} = e^{y} \cdot e^{-2x}$$

$$e^{-y}\,dy = e^{-2x}\,dx$$

$$\int e^{-y}\,dy = \int e^{-2x}\,dx$$

$$-e^{-y} = -\frac{1}{2}e^{-2x} + C$$

$$e^{-y} = \frac{1}{2}e^{-2x} + C$$

$$-y = \ln\left(\frac{1}{2}e^{-2x} + C\right)$$

$$\boxed{y = -\ln\left(\frac{1}{2}e^{-2x} + C\right)}$$

Ex 2. Solve the IVP $y' = 3\ln t$, $y(1) = 7$

$$\frac{dy}{dt} = 3\ln t \text{, so } dy = 3\ln t\, dt$$

$$y = \int dy = \int 3\ln t\, dt$$

Try integration by parts

$u = \ln t$ $\quad$ $dv = 3dt$

$du = \frac{1}{t}dt$ $\quad$ $v = 3t$

$$y = 3t\ln t - \int 3t \cdot \frac{1}{t}\, dt$$

$$y = 3t\ln t - 3t + C$$

$$7 = y(1) = 3(1)\ln(1) - 3(1) + C$$

$$7 = -3 + C \text{, } C = 10$$

$$\boxed{y(t) = 3t\ln t - 3t + 10}$$

Growth and Decay

Growth and decay are accurately modeled through differential equations in which the rate of change of a function is proportional to the function.

If y' is (directly) proportional to y, then $y' = ky$ for some constant k.

If y' is inversely proportional to y, then $y' = \frac{k}{y}$ for some constant k.

If y' is jointly proportional to y and w, then $y' = kyw$, for some constant k.

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(skip?)
Ex 3. The rate of change of the number of turtles $N(t)$ in a population is directly proportional to $400-N(t)$, where t is measured in years. When $t=0$, the population is 300. When $t=3$, the population is 380. Find the population when $t=4$. (Round to whole number)

$$\frac{dN}{dt} = k(400-N)$$

$$\int \frac{dN}{400-N} = \int k\,dt \qquad u = 400-N, \; du = -dN$$

$$-\ln|400-N| = kt + C$$

$$\ln(400-N) = -kt + C$$

$$|400-N| = e^{-kt+C} = e^{-kt}e^{C} = Ce^{-kt}$$

$$400-N = Ce^{-kt}$$

$$N = Ce^{-kt} + 400$$

$$300 = N(0) = C + 400, \text{ so } C = -100$$

$$N = -100e^{-kt} + 400$$

$$380 = N(3) = -100e^{-3k} + 400$$

$$-20 = -100e^{-3k}$$

$$\tfrac{1}{5} = e^{-3k} \Rightarrow -3k = \ln(\tfrac{1}{5}) \Rightarrow k = -\tfrac{1}{3}\ln(\tfrac{1}{5})$$

$$N(t) = -100e^{\frac{1}{3}\ln(\frac{1}{5})t} + 400$$

$$N(4) = -100e^{\frac{1}{3}\ln(\frac{1}{5})\cdot 4} + 400$$

$$\approx \boxed{388 \text{ turtles}}$$

Ex 4. A radioactive element decays proportionally to its amount. Its half-life is 4 years. If it starts with 30 mg, how much is left in 7 years? (Round to 2 decimal places)

$$\frac{dA}{dt} = kA$$

$$\int \frac{dA}{A} = \int k\,dt$$

$$\ln|A| = kt + C$$

$$A = Ce^{kt}$$

$$30 = A(0) = Ce^0 = C$$

$$A(t) = 30e^{kt}$$

$$15 = 30e^{4k}$$

$$\tfrac{1}{2} = e^{4k}$$

$$4k = \ln(\tfrac{1}{2}) \Rightarrow k = \tfrac{1}{4}\ln(\tfrac{1}{2})$$

$$A(t) = 30e^{\frac{1}{4}\ln(\frac{1}{2})t}$$

$$A(7) = 30e^{\frac{1}{4}\ln(\frac{1}{2})\cdot 7} \approx \boxed{8.92 \text{ mg}}$$

Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between the object's temperature and the temperature of the surrounding area, i.e.,

$$\frac{dT}{dt} = k(T - T_s)$$

where T_s is the surrounding temperature and $T(t)$ is temperature as a function of time.

Solving...

$$\int \frac{dT}{T - T_s} = \int k\,dt$$

$$\ln|T - T_s| = kt + C$$

$$T - T_s = Ce^{kt}$$

$$\boxed{T(t) = T_s + Ce^{kt}}$$

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Ex 5. After 6 minutes, the temperature of a cup of hot cocoa has decreased to 60°C from 100°C. The room has a temperature of 22°C. How much longer will it take to cool down to 40°C? Round to the nearest hundredth.

Newton's Law of Cooling gives

$$T(t) = T_s + Ce^{kt}$$

$$T_s = 22, \quad T(0) = 100, \quad T(6) = 60$$

$$T(t) = 22 + Ce^{kt}$$

$$100 = T(0) = 22 + Ce^0 = 22 + C$$

$$C = 78$$

$$T(t) = 22 + 78e^{kt}$$

$$60 = T(6) = 22 + 78e^{6k}$$

$$\frac{38}{78} = e^{6k}$$

$$k = \frac{1}{6}\ln\left(\frac{19}{39}\right)$$

So $T(t) = 22 + 78e^{\frac{1}{6}\ln\left(\frac{19}{39}\right)t}$

Set $40 = 22 + 78e^{\frac{1}{6}\ln\left(\frac{19}{39}\right)t}$

$$18 = 78e^{\frac{1}{6}\ln\left(\frac{19}{39}\right)t}$$

$$\frac{3}{13} = e^{\frac{1}{6}\ln\left(\frac{19}{39}\right)t}$$

$$\ln\left(\frac{3}{13}\right) = \frac{1}{6}\ln\left(\frac{19}{39}\right)t$$

$$t = \frac{6\ln\left(\frac{3}{13}\right)}{\ln\left(\frac{19}{39}\right)} \approx 12.23 \text{ minutes}$$

Asked for how much longer (so 6 minutes already past)

$$12.23 - 6 = \boxed{6.23 \text{ minutes}}$$