MA 16020
Lesson 8
Separable Equations (Part 2)

Ex 1. A 600 gallon tank initially contains 400 gallons of pure distilled water. Brine containing 3 pounds of salt per gallon flows into the tank at a rate of 4 gallons per minute, and the well-stirred mixture flows out at a rate of 1 gallon per minute. Set up a differential equation representing this scenario.

Let $A(t)$ represent the amount of salt (in lbs) in the tank at time t (in minutes)

Since we have rates of flow, we can figure out the rate of change of A (i.e., $\frac{dA}{dt}$)

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate of amount of salt} \\ \text{flowing in} \\ \text{(lbs/min)}\end{pmatrix} - \begin{pmatrix}\text{rate of amount of salt} \\ \text{flowing out} \\ \text{(lbs/min)}\end{pmatrix}$$

$$= \begin{pmatrix}\text{concentration} \\ \text{flowing in} \\ \text{(lbs/gal)}\end{pmatrix}\begin{pmatrix}\text{rate} \\ \text{flowing in} \\ \text{(gal/min)}\end{pmatrix} - \begin{pmatrix}\text{concentration} \\ \text{flowing out} \\ \text{(lbs/gal)}\end{pmatrix}\begin{pmatrix}\text{rate} \\ \text{flowing out} \\ \text{(gal/min)}\end{pmatrix}$$

As units suggest, concentration is $\frac{\text{amount}}{\text{volume}}$

$$= \left(\frac{3 \text{ lbs}}{\text{gal}}\right)\cdot\left(\frac{4 \text{ gal}}{\text{min}}\right) - \left(\frac{\text{lbs in tank}}{\text{volume in tank}}\right)\cdot\left(\frac{1 \text{ gal}}{\text{min}}\right)$$

remember! A = amount (lbs) in the tank at time t (and this changes)

Also, notice that the volume in the tank changes! It starts with 400 gallons and every minute, 4 gallons come in and 1 gallon leaves, so 3 gallon increase.

$V(t) = 400 + 3t$ gallons at time t

$$\boxed{\frac{dA}{dt} = 12 - \frac{A}{400+3t}}$$

← This is not separable, so we can't solve it yet.

To see an example we can solve, see example 5.

Ex 2. The rate of change in the number of miles of road cleared per hour by a snowplow is inversely proportional to the depth of the snow. Given that 30 miles per hour are cleared when the depth is 2 inches and 16 miles per hour are cleared when the depth is 8 inches, how many miles of road will be cleared per hour when the depth of snow is 12 inches? (Round 2 decimals)

M = miles cleared per hour
s = depth of snow in inches

$$\frac{dM}{ds} = \frac{k}{s} \Rightarrow dM = \frac{k}{s}\,ds$$
$$\Rightarrow M = k\ln|s| + C$$

$$\begin{cases} 30 = k\ln(2) + C \\ 16 = k\ln(8) + C \end{cases} \Rightarrow \begin{array}{r} 30 = k\ln(2) + C \\ -16 = -k\ln(8) - C \\ \hline 14 = k\ln(2) - k\ln(8) \end{array}$$

$$14 = k(\ln(2) - \ln(8))$$
$$14 = k\ln\left(\tfrac{2}{8}\right) = k\ln\left(\tfrac{1}{4}\right)$$

so $k = \dfrac{14}{\ln(\frac{1}{4})}$

$16 = \dfrac{14}{\ln(\frac{1}{4})}\ln(8) + C$, so $C = 16 - \dfrac{14\ln(8)}{\ln(\frac{1}{4})}$

$$M = \frac{14}{\ln(\frac{1}{4})}\ln(s) + 16 - \frac{14\ln(8)}{\ln(\frac{1}{4})}$$

$$M(12) = \frac{14}{\ln(\frac{1}{4})}\ln(12) + 16 - \frac{14\ln(8)}{\ln(\frac{1}{4})}$$

$\approx$ **11.91 miles each hour**

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Ex 3. A substance is being converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time t. Initially, 38 grams of the first substance was present, and 1 hour later 15 grams remained. What is the amount of the first substance remaining after 5 hours? (Round to 2)

$A(t)$ = amount of first substance after t hours

$$\frac{dA}{dt} = kA^2$$

$$\int A^{-2}\,dA = \int k\,dt$$

$$-A^{-1} = kt + C$$

$$\frac{1}{A} = -kt + C$$

$$A = \frac{1}{-kt+C}$$

when $t=0$, $A=38$

$$38 = \frac{1}{C}, \text{ so } C = \frac{1}{38}$$

$$A = \frac{1}{-kt + \frac{1}{38}}$$

when $t=1$, $A=15$

$$15 = \frac{1}{-k + \frac{1}{38}}$$

$$\frac{1}{15} = -k + \frac{1}{38}$$

$$-k = \frac{1}{15} - \frac{1}{38} = \frac{23}{570}$$

$$A = \frac{1}{\frac{23}{570}t + \frac{1}{38}}$$

$$A(5) = \frac{1}{\frac{23}{570}(5) + \frac{1}{38}} \approx \boxed{4.38 \text{ grams}}$$

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Ex 4. Find the general solution

$$12x^2y' = y' + 3xe^{-y}$$

$$12x^2y' - y' = 3xe^{-y}$$

$$(12x^2-1)\frac{dy}{dx} = 3xe^{-y}$$

$$e^y\,dy = \frac{3x}{12x^2-1}\,dx$$

$u = 12x^2 - 1$

$du = 24x\,dx$

$\frac{1}{8}du = 3x\,dx$

$$\int e^y\,dy = \frac{1}{8}\int \frac{1}{u}\,du$$

$$e^y = \frac{1}{8}\ln|u| + C$$

$$e^y = \frac{1}{8}\ln|12x^2-1| + C$$

$$\boxed{y = \ln\left(\frac{1}{8}\ln|12x^2-1| + C\right)}$$

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Ex 5. A 600 gallon tank initially contains 400 gallons of pure distilled water. Brine containing 3 pounds of salt per gallon flows into the tank at a rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the same rate. Find the amount of salt in the tank after 20 minutes. (Round 2 decimal)

Let $S(t)$ be lbs of salt in tank after t minutes

$\frac{dS}{dt}$ = rate of change of lbs of salt in the tank after t minutes

= (rate of change of lbs of salt into tank) − (rate of change of lbs of salt out of tank)

units (lbs/min)

rate is (concentration of salt)·(rate of flow)

$$= \left(\frac{3\text{ lbs}}{\text{gal}} \cdot \frac{4\text{ gal}}{\text{min}}\right) - \left(\frac{S\text{ lbs}}{400\text{ gal}} \cdot \frac{4\text{ gal}}{\text{min}}\right)$$

$$= 12 - \frac{S}{100}$$

$$\frac{1}{12 - \frac{1}{100}S}\, dS = dt$$

$u = 12 - \frac{1}{100}S$, $du = -\frac{1}{100}\,dS \Rightarrow -100\,du = dS$

$$-100\int \frac{1}{u}\,du = \int dt$$

$$-100 \ln\left|12 - \frac{1}{100}S\right| = t + C$$

$$\ln\left|12 - \frac{1}{100}S\right| = -\frac{t}{100} + C$$

$$12 - \frac{1}{100}S = Ce^{-t/100}$$

$$-\frac{1}{100}S = Ce^{-t/100} - 12$$

$$S = Ce^{-t/100} + 1200$$

when $t = 0$, $S = 0$ (pure water), $C = -1200$

$$S(20) = -1200e^{-20/100} + 1200 \approx \boxed{217.52\text{ lbs}}$$