MA 16020
Lesson 9
First Order Linear Diff Eqs

A first order linear diff eq is a diff eq which can be written in the form

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$

functions of t

Leibniz noticed the LHS of such an equation looks like the result of the product rule.
For some function $u(t)$,

$$\frac{d}{dt}[u(t)\,y(t)] = u(t)\frac{dy}{dt} + y\frac{du}{dt}$$

Multiplying our first equation by $u(t)$, we get $u(t)\frac{dy}{dt} + u(t)P(t)\,y = u(t)Q(t)$

We seek $u(t)$ so $\underline{u(t)\frac{dy}{dt}} + u(t)P(t)y = \underline{u(t)\frac{dy}{dt}} + \frac{du}{dt}y$

$$\frac{du}{dt} = u\,p(t)$$

$$\int \frac{1}{u}\,du = \int p(t)\,dt$$

$$\ln|u| = \int p(t)\,dt + C$$

$$u = Ce^{\int p(t)\,dt}$$

Any constant C meets our needs, so choose $C = 1$.

$$u(t) = e^{\int p(t)\,dt}$$

$u(t)$ is called an integrating factor
$u(t)$ is defined in such a way that multiplying the diff eq by $u(t)$ gives $\frac{d}{dt}[u(t)\,y]$ on the LHS (which is easy to integrate)

Method of integrating factors

Given a first order linear diff eq,

1. Get it in the form $\frac{dy}{dt} + P(t)\,y = Q(t)$
   (no function by $\frac{dy}{dt}$)
2. Calculate the integrating factor $u(t) = e^{\int P(t)\,dt}$
3. Multiply both sides of the diff eq by $u(t)$
   to get $\frac{d}{dt}[u(t)\,y] = u(t)\,Q(t)$
4. Integrate both sides to get
   $u(t)\,y = \int u(t)\,Q(t)\,dt + C$
5. Solve for y by dividing by $u(t)$.

Ex 1. If $y' + (\tan x)\,y = 5\cos x$ for the interval $0 < x < \frac{\pi}{2}$, what is y?

1. Already in proper form. ✓
2. $u(x) = e^{\int \tan x\,dx}$. $\int \frac{\sin x}{\cos x}\,dx$, $u = \cos x$, $du = -\sin x\,dx$

$\int \tan x\,dx = \int -\frac{1}{u}\,du = -\ln|u| = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$

So $u(x) = e^{\ln|\sec x|} = |\sec x| = \sec x \quad (0 < x < \frac{\pi}{2})$

3. $\frac{d}{dx}[\sec x\, y] = 5\sec x\cos x = 5$
4. $\sec x\, y = 5x + C$
5. $\boxed{y = 5x\cos x + C\cos x}$

Check: $y' = -5x\sin x + 5\cos x - C\sin x$

plug into $y' + (\tan x)\,y = 5\cos x$

$(-5x\sin x + 5\cos x - C\sin x) + \tan x(5x\cos x + C\cos x) \stackrel{?}{=} 5\cos x$

$\cancel{-5x\sin x} + 5\cos x \cancel{- C\sin x} + \cancel{5x\sin x} + \cancel{C\sin x} \stackrel{\checkmark}{=} 5\cos x$

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Ex 2. Solve the initial value problem
$t^2 y' - t y = 3, \quad y(1) = 7$

1. Divide by t^2: ☆ $\frac{dy}{dt} - \frac{1}{t} y = \frac{3}{t^2}$ ☆
2. $u(t) = e^{\int -\frac{1}{t} dt} = e^{-\ln|t|} = e^{\ln|\frac{1}{t}|} = |\frac{1}{t}| = \frac{1}{t}$
(dealing with positive #'s here since $t = 1$ in initial condition)
3. $\frac{d}{dt}\left[\frac{1}{t} y\right] = \frac{3}{t^3} = 3t^{-3}$
4. $\frac{1}{t} y = \frac{-3}{2} t^{-2} + C$
5. $y = \frac{-3}{2t} + Ct$

Solve IVP $\quad 7 = -\frac{3}{2} + C, \quad C = \frac{17}{2}$
so $\boxed{y(t) = -\frac{3}{2t} + \frac{17}{2} t}$

Ex 3. Find the general solution
$(y - 300) \sin x \, dx - dy = 0$

1. $(y-300) \sin x - \frac{dy}{dx} = 0$
$-\frac{dy}{dx} + (\sin x) y - 300 \sin x = 0$
$-\frac{dy}{dx} + (\sin x) y = 300 \sin x$
$\frac{dy}{dx} - (\sin x) y = -300 \sin x$
2. $u(x) = e^{\int -\sin x \, dx} = e^{\cos x}$
3. $\frac{d}{dx}\left[e^{\cos x} y\right] = -300 \sin x \, e^{\cos x}$
4. $w = \cos x, \quad dw = -\sin x \, dx$
$\int -300 \sin x \, e^{\cos x} dx = 300 \int e^{w} dw = 300 e^{\cos x}$
$e^{\cos x} y = 300 e^{\cos x} + C$

5. $y = 300 + \frac{C}{e^{\cos x}}$

$\boxed{y(x) = 300 + C e^{-\cos x}}$

Ex 4. $t\frac{dy}{dt} + 8y = \frac{3}{t}$, $t > 0$

1. $\frac{dy}{dt} + \frac{8}{t}y = \frac{3}{t^2}$
2. $u(t) = e^{\int \frac{8}{t}\,dt} = e^{8\ln|t|} = e^{\ln|t^8|} = |t^8| = t^8$ $(t>0)$
3. $\frac{d}{dt}[t^8 y] = 3t^6$
4. $t^8 y = \frac{3}{7}t^7 + C$
5. $y = \frac{3}{7t} + \frac{C}{t^8}$

$$\boxed{y(t) = \frac{3}{7t} + \frac{C}{t^8}}$$