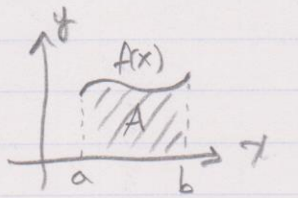
MA 16020

Lesson R(eview)

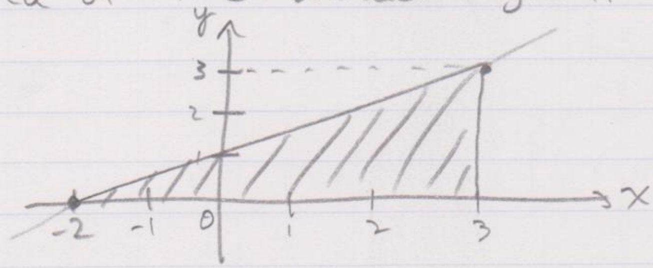
The Fundamental Theorem of Calculus

The area between a function $f(x)$ and the x-axis from $x=a$ to $x=b$ is called the (definite) integral of $f(x)$ from a to b, denoted $\int_a^b f(x)\,dx$.
The area above the x-axis is taken to be positive.
The area below the x-axis is taken to be negative.



$\int_a^b f(x)\,dx = A$, the area of the shaded region.

Example 1. Write the integral representing the area of the shaded region.



The curve $f(x)$ is a line passing through $(-2,0)$ and $(3,3)$.
To find the equation, we use point-slope form $y-y_0 = m(x-x_0)$.

$m = \frac{\Delta y}{\Delta x} = \frac{y_1-y_0}{x_1-x_0} = \frac{3-0}{3-(-2)} = \frac{3}{5}$

so $y-0 = \frac{3}{5}(x-(-2)) \implies y = \frac{3}{5}x + \frac{6}{5}$

So the area is $\boxed{\int_{-2}^{3} \left(\frac{3}{5}x + \frac{6}{5}\right) dx}$

Recall that a function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$.
Each function $f(x)$ has infinitely many antiderivatives and any two antiderivatives of $f(x)$ differ by a constant.
(They change the same way, so they must have the same shape, but could be shifted vertically.

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The <u>(indefinite) integral</u> of a function $f(x)$ is denoted by $\int f(x)\,dx$ (no bounds) and gives the most general antiderivative of $f(x)$. Generally, it is of the form $F(x)+C$.

<u>Example 2</u>. Find $\int \left(\frac{\sqrt[3]{x^2} + ex}{x} + 4\cos x - \frac{5}{x} + e^x \right) dx$

Can split up! $\int \frac{\sqrt[3]{x^2}}{x}\,dx + \int \frac{ex}{x}\,dx + 4\int \cos x\,dx - 5\int \frac{1}{x}\,dx + \int e^x\,dx$

$\sqrt[3]{x^2} = x^{2/3}$, so $\frac{\sqrt[3]{x^2}}{x} = \frac{x^{2/3}}{x} = x^{\frac{2}{3}-1} = x^{-1/3}$

$$\int x^{-1/3}\,dx + \int \underbrace{e}_{\text{constant}}\,dx + 4\int \cos x\,dx - 5\int \frac{1}{x}\,dx + \int e^x\,dx$$

$$\frac{x^{-1/3+1}}{-\frac{1}{3}+1} + ex + 4\sin x - 5\ln|x| + e^x + C$$

$$= \boxed{\frac{3}{2}x^{2/3} + ex + 4\sin x - 5\ln|x| + e^x + C}$$

You can always check your work by differentiating.

Don't forget the "+C".

If we're given an initial condition, we can find a specific antiderivative by solving for C.

<u>Example 3</u>. Given $y' = \frac{3}{x}$ and $y(e) = 7$, find $y(e^3)$.

$y = \int \frac{3}{x}\,dx = 3\ln|x| + C$ ← general antiderivative

But $y(e) = 7$, so

$7 = y(e) = 3\ln|e| + C = 3 + C$ (since $\ln(e) = 1$) $\Rightarrow C = 4$

$y(x) = 3\ln|x| + 4$

Then $y(e^3) = 3\ln|e^3| + 4 = 3 \cdot 3\ln(e) + 4 = 9 + 4 = \boxed{13}$

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Fundamental Theorem of Calculus

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x) = F(b) - F(a)$$

Ex 4. Compute $\int_0^1 \left(x\sqrt[3]{x^2} + x\right) dx$

$$\sqrt[3]{x^2} = x^{2/3}, \text{ so } x\sqrt[3]{x^2} = x \cdot x^{2/3} = x^{5/3}$$

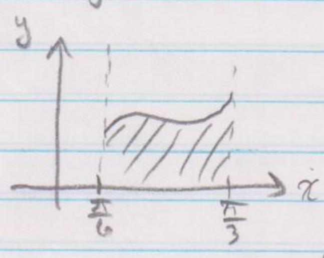
$$\int_0^1 \left(x^{5/3} + x\right) dx = \left(\tfrac{3}{8}x^{8/3} + \tfrac{1}{2}x^2\right)\Big|_0^1$$

$$= \left(\tfrac{3}{8}(1)^{8/3} + \tfrac{1}{2}(1)^2\right) - \left(\tfrac{3}{8}(0)^{8/3} + \tfrac{1}{2}(0)^2\right)$$

$$= \tfrac{3}{8} + \tfrac{1}{2} = \boxed{\tfrac{7}{8}}$$

Ex 5. Find the area of the region bounded by the graphs of

$$y = 3\sec^2 x + \sin x, \quad y = 0, \quad x = \tfrac{\pi}{6}, \quad x = \tfrac{\pi}{3}$$



This area is simply

$$\int_{\pi/6}^{\pi/3} \left(3\sec^2 x + \sin x\right) dx$$

$$= \left(3\tan x - \cos x\right)\Big|_{\pi/6}^{\pi/3}$$

$$= \left(3\tan(\tfrac{\pi}{3}) - \cos(\tfrac{\pi}{3})\right) - \left(3\tan(\tfrac{\pi}{6}) - \cos(\tfrac{\pi}{6})\right)$$

$$= \left(3\sqrt{3} - \tfrac{1}{2}\right) - \left(3\tfrac{\sqrt{3}}{3} - \tfrac{\sqrt{3}}{2}\right)$$

$$= 3\sqrt{3} - \tfrac{1}{2} - \sqrt{3} + \tfrac{\sqrt{3}}{2}$$

$$= \boxed{\frac{5\sqrt{3} - 1}{2}}$$

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Ex 6. At 3pm, there is no snow on the ground and snow begins to fall. The rate of change of the amount of snow is given by $A'(t) = 3t+2$ mm/hr, where t is measured in hours after 3pm.

(a) How much snow falls between 4pm and 6pm?

4pm is $t=1$, 6pm is $t=3$

So want $A(3) - A(1) = \int_1^3 A'(t)\,dt$

$$\int_1^3 (3t+2)\,dt = \left(\frac{3}{2}t^2 + 2t\right)\Big|_1^3$$

$$= \left(\frac{3}{2}(3)^2 + 2(3)\right) - \left(\frac{3}{2}(1)^2 + 2(1)\right)$$

$$= \frac{27}{2} + 6 - \frac{3}{2} - 2$$

$$= \boxed{16 \text{ mm}}$$

(b) After how many hours after 3pm will there be 3 cm of snow? Round to the nearest tenth.

3 cm = 30 mm

The amount of snow b hours after 3pm is

$$\int_0^b A'(t)\,dt \overset{\text{want}}{=} 30$$

$$\int_0^b (3t+2)\,dt = \left(\frac{3}{2}t^2 + 2t\right)\Big|_0^b = \frac{3}{2}b^2 + 2b$$

want $\frac{3}{2}b^2 + 2b = 30 \iff \frac{3}{2}b^2 + 2b - 30 = 0$

$$b = \frac{-2 \pm \sqrt{4 - 4\left(\frac{3}{2}\right)(-30)}}{2\left(\frac{3}{2}\right)} = \frac{-2 \pm \sqrt{184}}{3}$$

$b \approx 3.9$ or -5.2

negative answer doesn't make sense

so $\boxed{3.9 \text{ hours}}$