Math 266 **Final Exam Study Guide** **Spring 2019**

Information:
The exam will be 20 questions, all multiple choice. You will have 2 hours to complete the exam. The exam is cumulative. In addition to the information here, you can look at both study guides on the course website, as well as past final exams for MA 266. This study guide may not be an exhaustive list of topics.

1 Exam 1 Material

Direction/Slope Fields:

terminology to know: direction field, slope field (same thing as direction field), equilibrium solution

You should be able to: select which direction field corresponds to a given differential equation, select which differential equation produces a given direction field, use a direction field to describe the behavior of solutions

Practice Problems:
Look in the textbook.

Classification of differential equations:

terminology to know: order of a differential equation, ordinary vs. partial diff eq, linear vs. nonlinear diff eq

You should be able to: identify whether a given differential equation is ordinary or partial, linear or nonlinear, and be able to identify the order of the diff eq.

Practice Problems:
• For each of the differential equations below, determine whether it is partial or ordinary, linear or nonlinear, and state the order.

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+2y=\sin t$$
$$\frac{d^2y}{dt^2}+\sin(t+y)=\sin t$$
$$\frac{d^3y}{dt^3}+t\frac{dy}{dt}+\left(\cos^2 t\right)y=t^3$$
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

Solutions of differential equations:

terminology to know: solution to a differential equation, general solution, initial condition, initial value problem, solution to an initial value problem, integral curve

You should be able to: identify whether a given differential equation is ordinary or partial, linear or nonlinear, and be able to identify the order of the diff eq.

Practice Problems:
- For what value(s) of A, if any, will $y = Ate^{-2t}$ be a solution of the differential equation $2y' + 4y = 3e^{-2t}$? For what value(s) of B, if any, will $y = Be^{-2t}$ be a solution?

Solving First Order Diff Eqs:

terminology to know: first order linear differential equation, integrating factor, integrating factor method, separable differential equation, explicit solution, implicit solution, homogeneous differential equation (y/x version), Bernoulli differential equation, exact differential equation

You should be able to: identify and solve any of the types of differential equations listed above, finding the general solution or a solution to an initial value problem, find the domain of a solution to a separable equation.

Practice Problems:
For each of the differential equations below, find the explicit solution, if possible. If it is not possible to solve explicity, give an implicit solution. If the diff eq is separable, give the domain(s) of the solution(s).
- $t^2y' + 2ty - y^3 = 0, t > 0$
- $y' = \dfrac{6y + 2x}{3y^2 - 6x}$
- $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$
- $\dfrac{dy}{dx} = \dfrac{x^2}{y}, y(0) = -\dfrac{2}{3}$
- $t^3y' + 4t^2y = e^{-t}, y(-1) = 0, t < 0$

Substitutions:

You should be able to: solve a differential equation by making a substitution

Practice Problems:
- Using the substitution $u(x) = y + x$, solve the diff eq $\dfrac{dy}{dx} = (y + x)^2$.
- Using the substitution $u(x) = y^3$, solve the diff eq $y^2\dfrac{dy}{dx} + \dfrac{y^3}{x} = \dfrac{2}{x^2}, x > 0$.

Mathematical Models for First Order Diff Eqs:

terminology to know: mathematical model, gravity, air resistance, concentration (of some chemical in another chemical), terminal velocity (of a falling body), Newton's Law of Cooling

You should be able to: set up initial value problems representing a scenario like above, solve simple application problems of the type above

Practice Problems:

- Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred mixture flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.
- A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 2 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank the instant it begins to overflow.
- A ball with mass 0.15 kg is thrown upward with an initial velocity 20 m/s from the roof of a building 30 m high. Assume there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where v is the velocity measured in m/s. Find the time it takes for the ball to hit the ground.
- A ball with mass 0.15 kg is thrown upward with an initial velocity 20 m/s from the roof of a building 30 m high. Assume there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where v is the velocity measured in m/s. Write a differential equation (or differential equations) modeling this scenario.
- A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft. Assume that the force of air resistance, which is directed opposite to velocity, is of magnitude $0.75\,|v|$, where v is measured in ft/s. What is the terminal velocity of the skydiver (assume they do not open their parachute).
- Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

Existence and Uniqueness for First Order Diff Eqs:

terminology to know: existence and uniqueness of a solution

You should be able to: understand and be able to apply Theorems 2.4.1 and 2.4.2 about the existence and uniqueness of solutions to initial value problems.

Practice Problems:

- Determine (without solving the diff eq) the largest interval on which a solution is guaranteed to exist: $(4-t^2)\,y' + 2ty = 3t^2$, $y(1) = -3$.
- State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied for $\dfrac{dy}{dt} = \dfrac{(\cot t)\,y}{3y - y^2}$.
- Even though the hypotheses of Theorem 2.4.2 are satisfied on the entire ty-plane for $y' = y^2 + 6y, y(0) = 2$, show (by finding the solution and observing its domain) that the domain is smaller than the interval $-\infty < t < \infty$.

Autonomous Equations and Population Dynamics:

terminology to know: autonomous diff eq, equilibrium solution, critical point, asymptotically stable, unstable, semistable, $f(y)$ vs. y graph, phase line, carrying capacity, threshold

You should be able to: graph f(y) vs. y, graph the phase line, find equilibrium solutions, classify equilibrium solutions as asymptotically stable, unstable, or semistable, draw a qualitatively accurate sketch of several integral curves based on the phase line and equilibrium solutions, be able to talk about the carrying capacity and threshold of a population which can be modeled by an autonomous equation.

Practice Problems:

- Draw the phase plane and several qualitatively accurate sketches of solutions for the differential equation $\dfrac{dy}{dt} = y\,(1-y^2)$. Determine the equilibrium solutions and state whether each is asymptotically stable, unstable, or semistable.
- Suppose a population is modeled by $\dfrac{dp}{dt} = p^2\,(5-p)\,(p-12)$, where p is measured in thousands. What are the threshold and carrying capacity of the population?

Euler's Method:

You should be able to: Understand how Euler's method works, use Euler's method to do some simple estimations of solutions to initial value problems

Practice Problems:

- Use Euler's method with step size $h = 1/2$ to estimate $y(1)$ for the intial value problem $y' = y + \cos(\pi t)\,, y(0) = 1$

Analysis of solutions:

You should be able to: Describe the behavior of solutions to a differential equation as $t \to \infty$, finding maxima and minima of solutions, finding critical initial values (values of a so that if $y(t_0) < a$, then the solution behaves in a very different way from if $y(t_0) > a$)

Practice Problems:
- Find the critical value a_0 of a for which behavior changes qualitatively. Then describe the behavior of solutions to the following IVP as $t \to \infty$ (depending on the value of a).

$$y' - \frac{1}{2}y = 2\cos(t)\,, y(0) = a$$

2 Exam 2 Material

<u>2nd order linear homogeneous differential equations with constant coefficients</u>:

terminology to know: homogeneous, characteristic polynomial, characteristic equation

You should be able to: find the general solution of a 2nd order linear homogeneous diff eq with constant coefficients, solve IVPs of this sort, analyze solutions to differential equations of this type (understand behavior), find critical values of an initial condition which affect the behavior of solutions, understand what roots the characteristic equation could have to give solutions of certain behavior.

Practice Problems:
- Find the general solution of $y'' - 6y' + 9y = 0$.
- Find the general solution of $y'' - 6y' + 8y = 0$.
- Find the general solution of $y'' - 4y' + 13y = 0$.
- Find the general solution of $y'' + 2y = 0$.
- Solve the IVP $4y'' - y = 0, y(0) = 2, y'(0) = \beta$ and find the value of β so the solution approaches 0 as $t \to \infty$.
- Solve the IVP $9y'' + 12y' + 4y = 0, y(0) = a > 0, y'(0) = 1$ and find the value of a that separates solutions that become negative from those that are always positive.
- Find values of a, b, and c so that the diff eq $ay'' + by' + cy = 0$ has solutions whose behavior exhibits growing oscillations.
- Find values of a, b, and c so that the diff eq $ay'' + by' + cy = 0$ has solutions which tend to 0 as $t \to \infty$ but also have a local maximum or minimum.

<u>Theory of 2nd order linear differential equations</u>:

terminology to know: existence and uniqueness, principle of superposition, fundamental set of solutions, Wronskian

You should be able to: find the largest open intervals on which a unique solution to an IVP exists, apply the principle of superposition, know what a fundamental set is, determine whether a set of solutions is a fundamental set, understand the connections between fundamental sets and general solutions, compute the Wronskian.

Practice Problems:
- Find the largest open interval on which the IVP is guaranteed to have a unique solution: $(x-2)y'' + y' + (x-2)(\tan x)y = 0$, $y(3) = 1$, $y'(3) = 2$
- If $e^{t/2}$ and $e^{-t/2}$ are solutions to a differential equation, prove that $\sinh(t/2)$ is a solution as well.
- Suppose e^t and te^t are solutions to a second order linear diff eq. Show that $c_1e^t + c_2te^t$ includes all solutions to the differential equation.

Reduction of Order:

terminology to know: method of reduction of order

You should be able to: use the method of reduction of order to find the general solution to a 2nd order linear diff eq (with nonconstant coefficients) given one solution, know when to use the method of reduction of order (when one solution of a 2nd order diff eq is given to you)

Practice Problems:
- Find the general solution of $t^2y'' + 3ty' + y = 0, t > 0$ given that $y_1 = t^{-1}$ is a solution.

Nonhomogeneous diff eqs:

terminology to know: nonhomogeneous, nonhomogeneous part, complementary solution, particular solution, undetermined coefficients, variation of parameters

You should be able to: determine the form of a particular solution using the method of undetermined coefficients, find a particular solution/general solution using the method of undetermined coefficients, find a particular solution/general solution using the method of variation of parameters

Practice Problems:
- Find the form of a particular solution for the diff eq $y'' - 2y' - 3y = e^{3t} + 4t^3e^{-t} + 5\cos(2t)$
- Find the form of a particular solution for the diff eq $y'' - 2y' = 8t^3 + t + e^t\sin(t) + te^{-2t}$
- Find the general solution to the diff eq $y'' + 9y = \cos(2t)$.
- Find the general solution to $ty'' - (1+t)y' + y = t^2e^{2t}$, $t > 0$, given that $y_1 = 1 + t$ and $y_2 = e^t$ form a fundamental set of solutions to the corresponding homogeneous diff eq.

Mass-Spring Systems and Vibrations:

terminology to know: mass-spring system, damping coefficient, spring coefficient, Newtons, dynes, undamped vibrations, damped vibrations, free vibrations, forced vibrations, amplitude, period, natural frequency, phase shift, underdamped, critically damped, overdamped, quasi-frequency, quasi-period, forcing function, transient solution, steady state solution, beats/amplitude modulation, resonance

You should be able to: set up an IVP representing the motion of a mass-spring system whether it is undamped or damped and whether it is free or forced, keep units consistent, solve basic mass-spring system IVPs, convert $A\cos(\alpha t) + B\sin(\alpha t)$ to $R\cos(\omega_0 t - \delta)$ form, find the amplitude, period, natural frequency, and phase shift of an undamped free system, determine whether a free damped system is critically damped, overdamped, or underdamped, determine the value of the damping coefficient for which the system is critically damped, find the quasi-period and quasi-frequency of an underdamped free system, determine the transient and steady state solutions for a damped forced system, determine the frequency of the forcing function required for an undamped forced system to experience resonance, describe the conditions necessary to get beats.

Practice Problems:
• A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external foce of $10\sin(t/2)$ N (Newtons) and moves in a medium that imparts a viscuous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass. (Specificy the units of u.)
• Convert $3\cos 2t + 4\sin 2t$ into $R\cos(\omega_0 t - \delta)$ form.
• A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the natural frequency, period, phase, and amplitude of the resulting motion.
• A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attaches to a viscous damper with a damping constant of 400 dyn·s/cm. If the mass is pulled down an additional 2 cm and then released, determine the quasi-frequency and quasi-period of the resulting motion.
• A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with coefficient γ. Determine the value of γ for which the system is critically damped; be sure to give the units for γ. State all possible values (in interval form) of γ for which the system is underdamped; do the same for when the system is overdamped.
• Suppose a mass-spring system is represented by $u'' + u' + 1.25u = 3\cos t$ subject to the initial condition $u(0) = 2, u'(0) = 3$. Find the steady state and transient solutions. Is the motion asymptotic to the steady state or to the transient solution?
• Suppose a mass-spring system is represented by $u'' + u = 0.5\cos 0.8t, u(0) = 0, u'(0) = 0$. Does the system experience beats or resonance?
• Suppose a mass-spring system is represented by $u'' + u = 3\cos\omega t$. Find the value of ω for which the system experiences resonance.

Higher Order Linear Diff Eqs:

You should be able to: find the largest open interval on which an nth order linear IVP with constant coefficients has a unique solution, solve an nth order homogeneous linear diff eq with constant coefficients

Practice Problems:
- Determine the largest intervals on which solutions are sure to exist: $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$.
- Find the general solution: $2y''' - 4y'' - 2y' + 4y = 0$.
- Find the general solution: $y^{(6)} - y'' = 0$.
- Find the general solution: $y^{(4)} + 2y'' + y = 0$.

3 Post-Exam 2 Material

Undetermined Coefficients (nth order):

You should be able to: determine the form of a particular solution using the method of undetermined coefficients, find a particular solution using the method of undetermined coefficients.

Practice Problems:
- Find the form of a particular solution for the diff eq $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
- Find the form of a particular solution for the diff eq if the complementary solution is $c_1e^t + c_2e^{-t} + c_3 + c_4t + c_5t^2 + c_6\cos t + c_7\sin t$ and the nonhomogeneous part is $t^3e^t + \cos 2t + t^5 + 3t$
- Find the general solution to the diff eq $y^{(6)} + y^{(4)} = t$.

Laplace Transform:

terminology to know: Laplace transform, Heaviside step function, unit impulse function/Dirac delta function, convolution integral

You should be able to: know the definition of a Laplace transform, find the Laplace transform of a function in the time domain (using the table), find the inverse Laplace transform of a function in the frequency domain (using the table), rewrite a piecewise defined function in terms of step functions and vice versa, use the Laplace transform to solve initial value problems where the nonhomogeneous part is discontinuous (involving step functions or Dirac delta functions), use the Laplace transform to find solutions to IVPs in terms of convolution integrals, solve convolution integrals by using the Laplace transform.

Practice Problems:
- Find the inverse Laplace transform of the following functions:

$$F(s) = \frac{s}{s^2 + 3s - 4}$$

$$F(s) = \frac{3s}{s^2 - s - 6}$$

$$F(s) = \frac{2s-3}{s^2-4}$$

(if your answer to the third problem here is written explicitly in terms of exponential functions, do it again without writing your answer explicitly in terms of exponential functions! hint: check the Laplace table carefully)

• Convert the piecewise defined function into a function which is written in terms of step functions (assume the domain is $t \geq 0$):

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t^2 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 5 \\ 0 & t \geq 5 \end{cases}$$

• Write the function as a piecewise defined function: $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

• Solve the initial value problem

$$y'' + 2y' + 2y = \begin{cases} 1 & \pi \leq t < 2\pi \\ 0 & 0 \leq t < \pi \text{ and } t \geq 2\pi \end{cases}, \qquad y(0) = 0, y'(0) = 1$$

• Solve the initial value problem

$$y'' + y' + \frac{5}{4}y = \begin{cases} \sin t & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases}, \qquad y(0) = 0, y'(0) = 0$$

• Solve the IVP $y'' + 2y' + 2y = \delta(t-\pi), y(0) = 1, y'(0) = 0$

• Solve the IVP $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), y(0) = 0, y'(0) = 0$

• Evaluate the integral by using the Laplace transform: $\int_0^t e^{t-\tau} \cos\tau \, d\tau$

• Find the inverse Laplace transform in terms of convolution integrals

$$F(s) = \frac{1}{s^4(s^2+1)}$$

• Find the solution in terms of convolution integrals: $y''+3y'+2y = \cos\alpha t, y(0) = 1, y'(0) = 0$

General Idea of Systems of Diff Eqs:

You should be able to: convert 2nd order diff eqs into a system of 1st order diff eqs, convert a system of 1st order diff eqs into a 2nd order diff eq, set up a system of differential equations representing a physical scenario (interconnected tanks)

Practice Problems:
• Convert the 2nd order diff eq into a system of first order diff eqs (give initial conditions): $u'' + 0.25u' + 4u = 2\cos 3t, u(0) = 1, u'(0) = -2$
• Convert the system of first order diff eqs into a second order diff eq in the variable x_1 (make sure to give initial conditions!):

$$\begin{aligned} x_1' &= x_1 - 2x_2, \quad x_1(0) = -1 \\ x_2' &= 3x_1 - 4x_2, \quad x_2(0) = 2 \end{aligned}$$

• Convert the above system of first order diff eqs into a second order diff eq in the variable x_2 (make sure to give initial conditions!).
• Suppose there are two interconnected tanks. Tank 1 initially holds 30 gallons of brine (with a total of 5 lb of salt), and Tank 2 initially holds 40 gallons of pure water. Brine containing 2 lb of salt per gallon begins flowing into Tank 1 at a rate of 4 gal/min. The well-stirred mixture flows from Tank 1 into Tank 2 at a rate of 3 gal/min. Brine containing 3 lb of salt per gallon flows from above Tank 2 into Tank 2 at a rate of 1 gal/min, and the well-stirred mixture in Tank 2 flows out at a rate of 4 gal/min. Write a system of differential equations modeling this scenario. Be sure to include initial conditions!

Solving Systems of Diff Eqs:

terminology to know: eigenvalue, eigenvector, system of differential equations, generalized eigenvector

You should be able to: find the general solution of a system of differential equations where the eigenvalues are distinct real numbers, complex numbers, or the eigenvalue is a repeated real number, solve IVPs as above, use the method of undetermined coefficients or variation of parameters to solve nonohomogeneous systems of differential equations

Practice Problems:
• For each of the following systems, find the general solution:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

Analyzing the Phase Plane/Origin of Systems:

terminology to know: phase plane, phase portrait, trajectory, node, saddle point, spiral point, center, improper node, asymptotically stable, stable, unstable

You should be able to: analyze the behavior of the phase portrait of a 2x2 system of differential equations (if 0 is not an eigenvalue), classify the origin as a node, saddle point, spiral point, center, or improper node, and whether the origin is asymptotically stable, stable, or unstable. Determine values of an entry of the coefficient matrix which produce certain behaviors in the phase portrait

Practice Problems:

- For each of the *homogeneous* systems in the practice problems for the previous section, classify the origin as a node, improper node, spiral point, or center, and classify the stability of the origin as stable, asymptotically stable, or unstable.
- Find the critical values of α where the qualitative nature of the phase portrait for the system changes, and describe how the phase portrait changes at those critical values:

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$$

- Find all values of α so that the origin is an unstable spiral point:

$$\mathbf{x}' = \begin{pmatrix} 0 & 8 \\ -2 & \alpha \end{pmatrix} \mathbf{x}$$