Eddie Price - MA 266, Lesson 11 (SP 19)

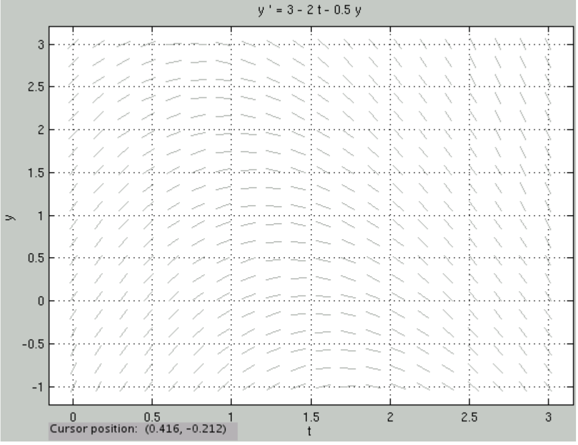
Euler's Method (Part 1)

So far in this course, we have seen some very special types of first order ODEs. We've seen methods to solve linear, separable, homogeneous, Bernoulli, and exact equations. Some of these methods don't even guarantee us an explicit solution, but rather give us a solution implicitly. These are very special types of first order ODEs. Given an arbitrary IVP, while we can employ the existence and uniqueness theorem to show that a solution (or even possibly a *unique* solution) exists, we may not be able to find a formula for such a solution - even implicitly! This is where numerical methods come into play.

Numerical methods are techniques which can be used to approximate solutions to IVPs numerically without ever expressly finding an equation representing the solution. While it is often handy to find actual equations representing solutions, in the real world (physics, engineering, biology, chemistry, etc.), we usually only need an approximation which is accurate to several decimal places.

Today, we are covering a particular technique developed by Swiss mathematician Leonhard Euler (1707-1783), which we call Euler's Method. Other techniques exist, but will not be discussed in this course.

We begin exploring Euler's method by looking at direction fields. Consider the direction field below.



We can imagine approximating solutions by a sort of "connect the dots" type method, where we connect the small tangent line segments. You probably already imagined doing things like this when we studied direction fields in lesson 1! And this is the basic idea behind Euler's Method: We approximate solutions by using small line segments based on the slope.

Euler's Method

Given an IVP $\frac{dy}{dt} = f\left(t, y\right), \, y\left(t_0\right) = y_0$, we take small steps, creating tangent lines along the way:

We begin by taking $f\left(t_0, y_0\right)$. This is a slope for a tangent line to the actual solution. We have a slope and a point, so we can form the line $y_{\text{tangent}} = y_0 + f\left(t_0, y_0\right) \cdot \left(t - t_0\right)$.

Then, given any fixed value $t = s$, we can approximate $y\left(s\right)$ by looking at $y_{\text{tangent}}\left(s\right)$. Of course, this approximation is only good when s is very close to t_0. This tangent line will likely get quite far off from the function as t moves away from t_0. As such, we break up our process into small tangent line segments.

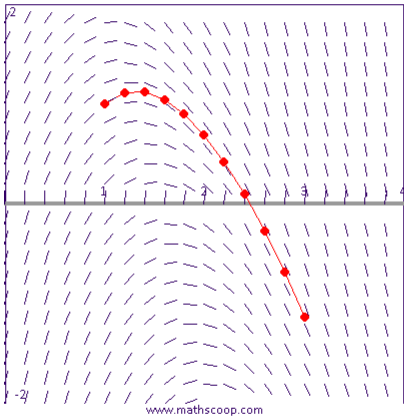
We are given time t_0 and solution value y_0. Choose another time t_1. We can form the line $y_{\text{tangent}} = y_0 + f\left(t_0, y_0\right) \cdot \left(t - t_0\right)$. We can then approximate $y\left(t_1\right)$ by taking $y_1 = y_0 + f\left(t_0, y_0\right) \cdot \left(t_1 - t_0\right)$. This gives us a point $\left(t_1, y_1\right)$ on this initial tangent line. We can then find the value of the slope field at this point $\left(t_1, y_1\right)$, and make another tangent line from this point to a point at t_2, etc.

Repeating this process, we obtain the equation $y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n)$ for any $n \geq 0$.

Often, we will make our process very regular, and increment t-values by a constant, which we will call h. This simplifies the above formula to

$$y(t_{n+1}) \approx y_{n+1} = y_n + f(t_n, y_n) \cdot h$$

Overall, this is what this process looks like, geometrically speaking:



The smaller that h is, the more accurate our approximations will be. But, for smaller h, the amount and difficulty of our computations grow.

Example 1. Use Euler's method with step size $h = 1/2$ to approximate $y\left(2\right)$ for the initial value problem

$$\frac{dy}{dt} = y + \sin\left(\pi t\right),\ y\left(0\right) = -2$$

We start by noticing $t_0 = 0$ and $y_0 = -2$.

Then $t_1 = t_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$

$$\begin{aligned} y\left(\frac{1}{2}\right) = y(t_1) \approx y_1 &= y_0 + f\left(t_0, y_0\right) \cdot h \\ &= -2 + \left(-2 + \sin\left(0\right)\right) \cdot \left(\frac{1}{2}\right) \\ &= -3 \end{aligned}$$

Then $t_2 = t_1 + h = \frac{1}{2} + \frac{1}{2} = 1$

$$\begin{aligned} y(1) = y(t_2) \approx y_2 &= y_1 + f\left(t_1, y_1\right) \cdot h \\ &= -3 + \left(-3 + \sin\left(\frac{\pi}{2}\right)\right) \cdot \left(\frac{1}{2}\right) \\ &= -4 \end{aligned}$$

Then $t_3 = t_2 + h = 1 + \frac{1}{2} = \frac{3}{2}$

$$\begin{aligned} y\left(\frac{3}{2}\right) = y(t_3) \approx y_3 &= y_2 + f\left(t_2, y_2\right) \cdot h \\ &= -4 + \left(-4 + \sin\left(\pi\right)\right) \cdot \left(\frac{1}{2}\right) \\ &= -6 \end{aligned}$$

Then $t_4 = t_3 + h = \frac{3}{2} + \frac{1}{2} = 2$

$$\begin{aligned} y(2) = y(t_4) \approx y_4 &= y_3 + f\left(t_3, y_3\right) \cdot h \\ &= -6 + \left(-6 + \sin\left(\frac{3\pi}{2}\right)\right) \cdot \left(\frac{1}{2}\right) \\ &= \frac{-19}{2} = -9.5 \end{aligned}$$

Our goal was to approximate $y\left(2\right)$, so we are done. $y(2) \approx -9.5$.

As we can see this process is quite tedious. You will be able to use MATLAB for your homework, but you should still understand and be able to compute simple calculations using Euler's method by hand. ***The information to use MATLAB for Euler's method can be found on the "eul" document on the course webpage***. If your version of MATLAB does not have eul.m and you cannot get it working on MATLAB, you may also use the online version on my webpage. Here are the basic instructions for using eul.m in MATLAB.

Basic procedure: Create a new .m file. In the code, make sure the following is typed in, but make sure to edit the parts between ⟨ and ⟩ (do not include ⟨ or ⟩ in the actual text).

```
function W=⟨function name⟩(t,y)
W=⟨insert y′ here⟩;
```

Then save the .m file as ⟨function name⟩.m

Next, go to the command line in MATLAB and type:

```
[t,y]=eul(’function name’,[initial t value, final t value],initial y value, step size);
```

Now, to display the approximation for y at each of the t-values, type into the command line:

```
[t,y]
```

Example 2. Use Euler's method with step size $h = 0.1$ to approximate $y(4)$ for the initial value problem

$$\frac{dy}{dt} = \frac{4 - ty}{1 + y^2},\ y(0) = 1$$

We start in MATLAB by creating a new function .m file. We enter the following information into MATLAB:

```
function W=fcn(t,y)
W=(4-t*y)/(1+y^2);
```

You then save the file as fcn.m

Returning to the MATLAB prompt, you type in the following command:

```
[t,y]=eul(’fcn’,[0,4],1,0.1);
```

Typing in the command [t,y] will then give you the values.

Example 3. (THIS IS PART OF YOUR HOMEWORK). Assume we are working with the differential equation $y' = 1 - t + y$.
Using the Euler formula, show that $y_k = (1+h)\, y_{k-1} + h - h t_{k-1}$, for $k = 1, 2, ...$

We begin with the fact that the Euler formula is $y_n = y_{n-1} + f_{n-1} \cdot (t_n - t_{n-1})$
Recall that $f_{n-1} = f\,(t_{n-1}, y_{n-1}) = 1 - t_{n-1} + y_{n-1}$ in this case.

Thus,

$$\begin{aligned} y_k &= y_{k-1} + (1 - t_{k-1} + y_{k-1})\,(t_k - t_{k-1}) \\ &= y_{k-1} + t_k - t_{k-1}t_k + y_{k-1}t_k - t_{k-1} + t_{k-1}^2 - y_{k-1}t_{k-1} \\ &= y_{k-1} + y_{k-1}t_k - y_{k-1}t_{k-1} + t_k - t_{k-1} - t_{k-1}t_k + t_{k-1}^2 \\ &= (1 + t_k - t_{k-1})\, y_{k-1} + (t_k - t_{k-1}) - (t_k - t_{k-1})\, t_{k-1} \\ &= (1+h)\, y_{k-1} + h - h t_{k-1} \end{aligned}$$

For this problem, even though you don't have to do part (c), you should use the results of part (c) when doing part (d).