

Lesson 13

19.1

Homogeneous Equations with Constant Coefficients (3.1)

Chapter 3 is about second order linear diff eqs. These are of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

Such an equation is called homogeneous if $g(t) \equiv 0$ (i.e., $g(t)$ is the zero function - for all t)

If $g(t) \neq 0$, then $g(t)$ is called the nonhomogeneous part of the equation.

In this section, we deal with homogeneous equations in which the coefficient functions are constants. i.e., of the form

$$ay'' + by' + cy = 0$$

We motivate how to solve such equations with the example $y'' - y = 0$.

It's not hard to see that $w_1(t) = e^t$ is a solution.

Also, $w_2(t) = e^{-t}$ is a solution.

And, for any constants c_1 and c_2 ,

$c_1 e^t + c_2 e^{-t}$ is a solution!

It seems natural that solutions of

$$ay'' + by' + cy = 0$$

Will then be of the form e^{rt} and if there is more than one, we get linear combinations.

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Assuming $y(t) = e^{rt}$ is a solution of

$$ay'' + by' + cy = 0, \quad \text{we see}$$

$$y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt}$$

$$ar^2 e^{rt} + br e^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2 + br + c) = 0$$

This is only possible when $ar^2 + br + c = 0$.

Given a diff eq of the form

$$ay'' + by' + cy = 0,$$

the equation $ar^2 + br + c = 0$

is called the characteristic equation of the diff eq.

The roots of the characteristic polynomial satisfy the diff. eq.

From algebra, we know the roots can be

- (i) distinct real numbers (this lesson)
- (ii) complex conjugates (lesson 15)
- (iii) a single repeated real number (lesson 16)

Thus, if given $ay'' + by' + cy = 0$,

1) Find the roots of the characteristic polynomial.

If they are real numbers $r_1 \neq r_2$, then

the general solution is of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where c_1 and c_2 are constants.

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Ex 1. Find the general solution to
 $y'' + y' - 12y = 0$

characteristic equation: $r^2 + r - 12 = 0$
 $(r+4)(r-3) = 0$

So $r_1 = -4$, $r_2 = 3$

$$y(t) = C_1 e^{-4t} + C_2 e^{3t}$$

For second order equations, we require two initial conditions to find a particular solution.

Ex 2. Solve the IVP

$$y'' + y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 3$$

$$r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0$$

$$r = 1 \text{ or } r = -2$$

$$y(t) = C_1 e^t + C_2 e^{-2t}$$

$$y'(t) = C_1 e^t - 2C_2 e^{-2t}$$

$$0 = y(0) = C_1 + C_2$$

$$3 = y'(0) = C_1 - 2C_2$$

Have system of equations $\begin{cases} C_1 + C_2 = 0 \\ C_1 - 2C_2 = 3 \end{cases}$

$$C_1 + C_2 = 0$$

$$-C_1 + 2C_2 = -3$$

$$3C_2 = -3$$

$$C_2 = -1$$

$$C_1 + (-1) = 0$$

$$C_1 = 1$$

$$y(t) = e^t - e^{-2t}$$

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Ex 3. Determine the behavior of $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ based on the values of r_1 and r_2 .

Notice that if r is negative, $\lim_{t \rightarrow \infty} c e^{rt} = 0$

If r is positive, $\lim_{t \rightarrow \infty} c e^{rt} = \pm \infty$ (depending on the sign of c)

If both r_1, r_2 are negative,
 $\lim_{t \rightarrow \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0$

If both r_1 and r_2 are positive,

$$\lim_{t \rightarrow \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty$$

depending on the signs of c_1, c_2 and whether $r_1 < r_2$ or $r_2 < r_1$.

If r_1 is positive and r_2 is non positive,

$$\lim_{t \rightarrow \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty$$

depending on the sign of c_1

Can have a max or min in this case!

If $r_1 = 0$ and r_2 is negative,

$$\lim_{t \rightarrow \infty} (c_1 e^0 + c_2 e^{r_2 t}) = c_1$$

These assume $c_1 \neq 0$ and $c_2 \neq 0$!

Ex 4. Solve the IVP

$$y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2.$$

Determine the value of α so the solution approaches zero as $t \rightarrow \infty$.

$$r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r=2 \text{ or } r=-1$$

$$y(t) = c_1 e^{2t} + c_2 e^{-t}, \quad y'(t) = 2c_1 e^{2t} - c_2 e^{-t}$$

$$\begin{cases} \alpha = c_1 + c_2 \\ 2 = 2c_1 - c_2 \end{cases}$$

$$\begin{cases} \alpha = c_1 + c_2 \\ 2 = 2c_1 - c_2 \end{cases}$$

$$2 + \alpha = 3c_1 \quad \text{so } c_1 = \frac{2 + \alpha}{3}$$

$$\alpha = \frac{2 + \alpha}{3} + c_2 \quad \text{so } c_2 = \frac{2 - 2\alpha}{3}$$

$$y(t) = \underbrace{\left(\frac{2 + \alpha}{3}\right)} e^{2t} + \left(\frac{2 - 2\alpha}{3}\right) e^{-t}$$

Since \rightarrow is positive and tends to ∞ as $t \rightarrow \infty$

Need $\frac{2 + \alpha}{3} = 0$, so $\boxed{\alpha = -2}$

Ex 5. Find the maximum value to the solution of $2y'' - 3y' + y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$

$$2r^2 - 3r + 1 = 0 \Rightarrow (2r-1)(r-1) = 0 \Rightarrow r = \frac{1}{2} \text{ or } r = 1$$

$$y(t) = c_1 e^t + c_2 e^{t/2}$$

$$\begin{cases} 2 = c_1 + c_2 \\ \frac{1}{2} = c_1 + \frac{c_2}{2} \end{cases} \rightarrow \begin{cases} 2 = c_1 + c_2 \\ -\frac{1}{2} = -c_1 - \frac{c_2}{2} \end{cases}$$

$$\frac{1}{2} = \frac{c_2}{2} \Rightarrow c_2 = 1$$

so $c_1 = -1$

$$y(t) = -e^t + e^{t/2}, \quad y'(t) = -e^t + \frac{1}{2}e^{t/2}$$

(continued)

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$$0 \stackrel{\text{set}}{=} -e^t + \frac{3}{2}e^{t/2}$$

Multiply by $e^{-t/2}$

$$0 = -e^{t/2} + \frac{3}{2}$$

$$-\frac{3}{2} = -e^{t/2}$$

$$\ln\left(\frac{3}{2}\right) = t/2$$

$$t = 2 \ln\left(\frac{3}{2}\right) = \ln\left(\frac{9}{4}\right)$$

$$y\left(\ln\left(\frac{9}{4}\right)\right) = -e^{\ln\left(\frac{9}{4}\right)} + 3e^{\ln\left(\frac{9}{4}\right)/2}$$

$$= -\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right)$$

$$= -\frac{9}{4} + \frac{9}{2}$$

$$= -\frac{9}{4} + \frac{18}{4}$$

$$= \boxed{\frac{9}{4}}$$