

Lesson 14

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Solutions of 2nd Order Linear Homogeneous Equations & The Wronskian (3.2)

Recall from linear algebra that a function $L: V \rightarrow W$ between vector spaces is called linear if $L(v_1 + v_2) = L(v_1) + L(v_2)$ and $L(cv) = cL(v)$.

A linear operator is a linear map for which our vector spaces have functions as their vectors.

Given a differential equation

$$y'' + py' + qy = 0,$$

we develop the linear operator

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Notice that $L[\phi]$ is a function!

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

ϕ is a solution to $y'' + py' + qy = 0$ if and only if $L[\phi] = 0$.

We now show that L is linear. Let ϕ and ψ be functions.

$$\begin{aligned} L[\phi + \psi] &= (\phi + \psi)'' + p(\phi + \psi)' + q(\phi + \psi) \\ &= \phi'' + \psi'' + p\phi' + p\psi' + q\phi + q\psi \\ &= \phi'' + p\phi' + q\phi + \psi'' + p\psi' + q\psi \\ &= L[\phi] + L[\psi] \end{aligned}$$

Let c be a constant

$$\begin{aligned} L[c\phi] &= (c\phi)'' + p(c\phi)' + q(c\phi) \\ &= c\phi'' + cp\phi' + cq\phi \\ &= c(\phi'' + p\phi' + q\phi) = cL[\phi] \end{aligned}$$

(This works because the diff eq is linear)

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The terminology of a linear operator is very tricky at first, but it will be extraordinarily useful in Chapter 6, and to prove the following theorem!

Theorem 3.2.2 (Principle of Superposition). If y_1 and y_2 are solutions to the linear equation $y'' + p(t)y' + q(t)y = 0$ then $c_1 y_1 + c_2 y_2$ is a solution as well for any constants c_1 and c_2 .

Proof. Use the linear operator $L[\phi] = \phi'' + p\phi' + q\phi$. Then $L[c_1 y_1 + c_2 y_2] = L[c_1 y_1] + L[c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = c_1(0) + c_2(0) = 0$. \square

Ex 1. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions to $y'' - y = 0$. Are $y_3(t) = \sinh t$ and $y_4(t) = \cosh t$ also solutions?

$$y_1' = e^t, y_1'' = e^t \text{ so } (e^t) - (e^t) = 0 \quad \checkmark$$

$$y_2' = -e^{-t}, y_2'' = e^{-t} \text{ so } (e^{-t}) - (e^{-t}) = 0 \quad \checkmark$$

By the Principle of Superposition, $c_1 e^t + c_2 e^{-t}$ is a solution for any c_1 and c_2 .

$$\sinh t = \frac{1}{2} e^t - \frac{1}{2} e^{-t} \quad (c_1 = \frac{1}{2}, c_2 = -\frac{1}{2})$$

$$\cosh t = \frac{1}{2} e^t + \frac{1}{2} e^{-t} \quad (c_1 = \frac{1}{2}, c_2 = \frac{1}{2})$$

So yes, these are also solutions!

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So we know that if two solutions exist, then any linear combination is also a solution. Can we establish that solutions exist? Yes!

Theorem 3.2.1 (Existence and Uniqueness Thm)

Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

If $p, q,$ and g are continuous on an interval I , then there exists a unique, twice-differentiable function $y = \phi(t)$ satisfying the IVP.

Ex 2. On what interval is her guaranteed to be a unique twice-differentiable solution to

$$(t+1)y'' + \ln(1-t)y' + y = \sqrt{t}, \quad y\left(\frac{1}{2}\right) = 1, \quad y'\left(\frac{1}{2}\right) = 2$$

$$y'' + \frac{\ln(1-t)}{(t+1)}y' + \frac{1}{t+1}y = \frac{\sqrt{t}}{t+1}$$

discontinuous when $t+1=0 \Rightarrow t=-1$

when $1-t \leq 0 \Rightarrow t \geq 1$

when $t < 0$

So continuous on $\boxed{0 \leq t < 1}$

We've addressed existence of solutions and how to form solutions from others. But how do we know we've found all solutions? How do we know, for example, that every solution to $y'' - y = 0$ can be written as

$c_1 e^t + c_2 e^{-t}$ for some constants c_1 and c_2 ?

To do this, we introduce the Wronskian.

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Given differentiable functions f and g , the Wronskian of f and g is the determinant

$$\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

Theorem 3.2.4 Suppose that y_1 and y_2 are solutions to $L[y] = y'' + p(t)y' + q(t)y = 0$. Then the family of solutions $y = c_1 y_1 + c_2 y_2$ includes every solution if and only if the Wronskian of y_1 and y_2 is not zero for some point t_0 .

(The proof comes from linear algebra — the solutions of $L[y] = 0$ are a vector space of dimension 2, and the Wronskian not being zero guarantees that $\{f, g\}$ is linearly independent, so it is a basis, so it spans the solution set)

If a set of solutions $\{y_1, y_2\}$ has Wronskian not equal to zero, we call it a fundamental set of solutions since $c_1 y_1 + c_2 y_2$ is the general solution, containing every solution.

Fundamental sets of solutions are not necessarily unique.

Ex 3. Show that $\{e^t, e^{-t}\}$ and $\{\sinh t, \cosh t\}$ are both fundamental sets of solutions to $y'' - y = 0$.

Already seen that all are solutions, so we just need to check the Wronskians.

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$$\begin{vmatrix} e^t & e^{-t} \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(e^{-t}) \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -e^t e^{-t} - e^t e^{-t}$$

$= -1 - 1 = -2$
 $-2 \neq 0$, so $\{e^t, e^{-t}\}$ is a fundamental set.

$$\begin{vmatrix} \sinh t & \cosh t \\ \frac{d}{dt}(\sinh t) & \frac{d}{dt}(\cosh t) \end{vmatrix} = \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix}$$

$$= \sinh^2 t - \cosh^2 t = -1$$

$-1 \neq 0$ so $\{\sinh t, \cosh t\}$ is a fundamental set.

Ex 4. Show that $\{e^t$ and $2e^t\}$ is not a fundamental set of solutions to $y'' - y = 0$.

$$\begin{vmatrix} e^t & 2e^t \\ -e^t & -2e^t \end{vmatrix} = -2e^{-2t} + 2e^{-2t} = 0$$

Wronskian is zero!

Ex 5. Suppose $y_1 = x$ and $y_2 = xe^x$ are solutions to $L[y] = 0$. Is $\{y_1, y_2\}$ a fundamental set?

$$\begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x^2 e^x + xe^x - xe^x = x^2 e^x$$

The Wronskian is not zero for some value of x (e.g. $x=1$), so it is a fundamental set.

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Ex 6, If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

$$3e^{4t} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g$$

$$e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

(First order linear ODE!)

$$g' - 2g = 3e^{2t}$$

$$\mu(t) = \exp\left(\int -2 dt\right) = e^{-2t}$$

$$\frac{d}{dt}[e^{-2t}g] = 3$$

$$e^{-2t}g = 3t + C$$

$$g(t) = 3te^{2t} + Ce^{2t}$$

for some constant C