

Lesson 15

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Complex Roots of the Characteristic Equation (3.3)

We saw in lesson 13 that if r_1 and r_2 are the roots of $ar^2 + br + c$, then the general solution of $ay'' + by' + cy = 0$ is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

What if these roots are complex conjugates

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu.$$

Then we have $c_1 e^{(\lambda + i\mu)t} + c_2 e^{(\lambda - i\mu)t}$

What does this even mean?

Leonhard Euler noticed a very interesting relationship when messing around with Maclaurin series.

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

What if we plug in $x = it$?

$$\begin{aligned} e^{it} &= 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \dots \\ &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \frac{t^6}{6!} - i\frac{t^7}{7!} + \dots \\ &= \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots\right) \\ &= \cos t + i \sin t \end{aligned}$$

So $e^{it} = \cos t + i \sin t$

We can use this formula to compute numbers raised to complex powers.

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Ex 1. Compute $e^{i\pi}$

$$e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 + i \cdot 0 = \boxed{-1}$$

Fact: Adding 1 to both sides gives
 $e^{i\pi} + 1 = 0$

which is regarded by many mathematicians as the most beautiful equation in mathematics.

Ex 2 - Compute 2^{3-4i}

$$\begin{aligned} 2^{3-4i} &= 2^3 \cdot 2^{-4i} \\ &= 8 \cdot 2^{-4i} \end{aligned}$$

Now, $2 = e^{\ln 2}$ so

$$\begin{aligned} 2^{3-4i} &= 8 \cdot (e^{\ln 2})^{-4i} = 8 \cdot e^{-4i \ln 2} \\ &= 8(\cos(-4 \ln 2) + i \sin(-4 \ln 2)) \\ &= \boxed{8 \cos(4 \ln 2) - 8i \sin(4 \ln 2)} \end{aligned}$$

(using $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$)

Ex 3. Compute $e^{(\lambda + i\mu)t}$

$$e^{(\lambda + i\mu)t} = e^{\lambda t} \cdot e^{i\mu t} = \boxed{e^{\lambda t} (\cos(\mu t) + i \sin(\mu t))}$$

$$\text{So } e^{(\lambda + i\mu)t} = \boxed{e^{\lambda t} \cos(\mu t) + i e^{\lambda t} \sin(\mu t)}$$

Using this fact, we can now make sense of the solution:

$$y(t) = k_1 e^{(\lambda + i\mu)t} + k_2 e^{(\lambda - i\mu)t}$$

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$$\text{So } y(t) = k_1 e^{\lambda t} \cos \mu t + k_1 i e^{\lambda t} \sin \mu t \\ + k_2 e^{\lambda t} \underbrace{\cos(-\mu t)}_{\cos(\mu t)} + k_2 i e^{\lambda t} \underbrace{\sin(-\mu t)}_{-\sin(\mu t)}$$

$$\text{So } y(t) = (k_1 + k_2) e^{\lambda t} \cos \mu t + i e^{\lambda t} (k_1 - k_2) \sin \mu t$$

$$y(t) = \underbrace{C_1}_{C_1} e^{\lambda t} \cos \mu t + i \underbrace{C_2}_{C_2} e^{\lambda t} \sin \mu t$$

Can we get this solution in terms of real numbers only?

Thm 3.2.6 If $u(t) + i v(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$, then $u(t)$ and $v(t)$ are solutions as well.

Proof. Use $L[\phi] = \phi'' + p(t)\phi' + q(t)\phi$

$$0 = L[u + iv] = L[u] + i L[v]$$

Real part is $L[u]$ which must equal 0

Imaginary part is $L[v]$ which must equal 0.

Thus, u and v are solutions. \square

So we know $e^{\lambda t} \cos \mu t$ and $e^{\lambda t} \sin \mu t$ are solutions. Can all solutions be written as a linear combination of these?

We check the Wronskian.

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$$\begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ -\mu e^{\lambda t} \sin \mu t + \lambda e^{\lambda t} \cos \mu t & \mu e^{\lambda t} \cos \mu t + \lambda e^{\lambda t} \sin \mu t \end{vmatrix}$$

$$= \mu e^{2\lambda t} \cos^2 \mu t + \lambda e^{2\lambda t} \cos(\mu t) \sin(\mu t) - (-\mu e^{2\lambda t} \sin^2 \mu t + \lambda e^{2\lambda t} \cos(\mu t) \sin(\mu t))$$

$$= \mu e^{2\lambda t} (\cos^2 \mu t + \sin^2 \mu t) = \mu e^{2\lambda t}$$

This Wronskian is nonzero whenever $\mu \neq 0$ (but if $\mu = 0$, then we would have repeated real roots, which we will address in lesson 16).

So, the Wronskian tells us that if $\lambda \pm i\mu$ are roots of the characteristic polynomial $ar^2 + br + c$ then the general solution to $ay'' + by' + cy = 0$

$$\text{is } y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

Ex 4. Solve the IVP

$$y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$$

$$r^2 + r + 1.25 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 5}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2}$$

$$r = -\frac{1}{2} \pm i$$

$$y(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$$

$$3 = y(0) = c_1 (1)(1) + c_2 (1)(0)$$

$$\text{so } c_1 = 3$$

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$$y(t) = 3e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$$

$$y'(t) = -3e^{-t/2} \sin t - \frac{3}{2}e^{-t/2} \cos t + c_2 e^{-t/2} \cos t - \frac{c_2}{2}e^{-t/2} \sin t$$

$$1 = y'(0) = -3(1)(0) - \frac{3}{2}(1)(1) + c_2(1)(1) - \frac{c_2}{2}(1)(0)$$

$$1 = -\frac{3}{2} + c_2 \quad \text{so} \quad c_2 = \frac{5}{2}$$

$$y(t) = 3e^{-t/2} \cos t + \frac{5}{2}e^{-t/2} \sin t$$

Graph this.

Notice it has decaying oscillation
(which makes sense since $\lim_{t \rightarrow \infty} y(t) = 0$
and $\sin t$ and $\cos t$ cause oscillation)

Ex 5. Solve the IVP

$$y'' + 16y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

$$r^2 + 16 = 0$$

$$r = 0 \pm 4i$$

$$y(t) = c_1 e^{0t} \cos(4t) + c_2 e^{0t} \sin(4t)$$

$$= c_1 \cos(4t) + c_2 \sin(4t)$$

$$1 = y(0) = c_1(1) + c_2(0), \quad \text{so} \quad c_1 = 1$$

$$y'(t) = -4 \sin(4t) + 4c_2 \cos(4t)$$

$$1 = y'(0) = 0 + 4c_2, \quad \text{so} \quad c_2 = \frac{1}{4}$$

$$y(t) = \cos(4t) + \frac{1}{4} \sin(4t)$$

Graph this.

Notice it has steady oscillation.

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Ex 6. Solve the IVP

$$y'' - 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1$$

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

$$r = 1 \pm i$$

$$y(t) = c_1 e^t \cos t + c_2 e^t \sin t$$

$$2 = y(0) = c_1(1)(1) + c_2(1)(0), \quad \text{so } c_1 = 2$$

$$y'(t) = -2e^t \sin t + 2e^t \cos t + c_2 e^t \cos t + c_2 e^t \sin t$$

$$1 = y'(0) = -2(1)(0) + 2(1)(1) + c_2(1)(1) + c_2(1)(0)$$

$$1 = -2 + c_2, \quad \text{so } c_2 = 3$$

$$y(t) = 2e^t \cos t + 3e^t \sin t$$

Graph this.

Note it has growing oscillation
(which makes sense since $e^t \rightarrow \infty$ as $t \rightarrow \infty$
and has $\cos t$ and $\sin t$ controlling oscillation)