

Lesson 16

pg. 1

Repeated Roots and Reduction of Order (3.4)

When the characteristic polynomial ar^2+br+c has a repeated real root, we run into some problems. We know that $y(t) = e^{rt}$ is a solution, but what about the other solution?

Jean d'Alembert (1717-1783) developed a technique for this situation. It is called the method of reduction of order. As the name suggests, using the solution, we use a clever trick to reduce the order of the differential equation to find another solution, which together form a fundamental set.

Suppose $ay''+by'+cy=0$ has a characteristic polynomial with repeated root r .
Notice $r = -\frac{b}{2a}$.

So we have $y_1 = e^{rt}$ as a solution.

We assume $y = v(t)y_1 = v(t)e^{rt}$ is the general solution, for some function $v(t)$.

$$\begin{aligned} \text{Then } y' &= r e^{rt} v + e^{rt} v' \text{ and} \\ y'' &= r e^{rt} v' + r^2 e^{rt} v + e^{rt} v'' + r e^{rt} v' \\ &= e^{rt} (v'' + 2r v' + r^2 v) \end{aligned}$$

$$\begin{aligned} \text{Then } a e^{rt} (v'' + 2r v' + r^2 v) + b e^{rt} (v' + r v) \\ + c e^{rt} v = 0 \end{aligned}$$

Lesson 16

pg. 2

Simplifying, we get...

$$\underbrace{e^{rt}}_{\text{never zero}} (av'' + 2arv' + ar^2v + bv' + brv + cv) = 0$$

$$av'' + \underbrace{(2ar+b)}_{=0 \text{ since } r = -\frac{b}{2a}} v' + \underbrace{(ar^2+br+c)}_{=0 \text{ since } r \text{ is a root}} v = 0$$

$$av'' = 0, \text{ or equivalently, } v'' = 0$$

Integrating, we get $v' = C_1$, then again, $v = C_1t + C_2$.

Since $y_2 = ve^{rt}$, we have $y = C_1te^{rt} + C_2e^{rt}$,
so te^{rt} is a solution as well.

We check that $\{e^{rt}, te^{rt}\}$ is a fundamental set.

$$\begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & rte^{rt} + e^{rt} \end{vmatrix} = rte^{2rt} + e^{2rt} - rte^{2rt} = e^{2rt} \neq 0$$

Thus, if the characteristic polynomial has a repeated root r , the general solution is

$$y(t) = c_1e^{rt} + c_2te^{rt}$$

Lesson 16

pg. 3

Ex 1. Find the general solution to
 $y'' - 4y' + 4y = 0$

$$r^2 - 4r + 4 = 0$$

$$(r - 2)^2 = 0$$

$$r = 2$$

so $y(t) = c_1 e^{2t} + c_2 t e^{2t}$

Ex 2. Find the general solution to
 $2y'' + 12y' + 18y = 0$

$$2r^2 + 12r + 18 = 0$$

$$r = \frac{-12 \pm \sqrt{144 - 144}}{4} = -3$$

so $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

The method of reduction of order works for other situations too!

Lesson 16

pg. 4

Method of Reduction of Order

Given a second order linear equation and one solution $y_1(t)$, assume $y(t) = v(t)y_1(t)$.

Then calculate y' and y'' .

Plug these into the diff eq,

After doing algebra, you will end up with a first order equation in v' . Solve this for v' .

Then integrate to get v . $v(t)y_1(t)$ is the other solution.

Ex 3. Find another solution to $t^2y'' + 2ty' - 2y = 0, t > 0; y_1(t) = t$.

Assume $y(t) = v(t)y_1(t) = v(t)t$

Then $y' = v(t) + tv'(t)$

$y'' = v'(t) + tv''(t) + v'(t) = tv''(t) + 2v'(t)$

$$t^2(tv'' + 2v') + 2t(v + tv') - 2(tv) = 0$$

$$t^3v'' + 2t^2v' + 2tv + 2t^2v' - 2tv = 0$$

$$t^3v'' + 4t^2v' = 0$$

This is first order in v' . Let $u = v'$, then $u' = v''$.

$$t^3u' + 4t^2u = 0$$

$$u' + \frac{4}{t}u = 0$$

$$u(t) = \exp\left(\int \frac{4}{t} dt\right) = e^{4 \ln t} = t^4$$

$$\frac{d}{dt}[t^4u] = 0$$

$$t^4u = k_1$$

$$v' = u = k_1 t^{-4}$$

$$\text{So } v = -\frac{k_1}{3}t^{-3} + k_2$$

Lesson 16

pg. 5

Thus $y(t) = v(t) \cdot t = \overset{\text{constant}}{\frac{-K_1}{3}} t^{-2} + \overset{y_1}{K_2 t}$

Hence, the other solution is just t^{-2} .

Check the Wronskian to be sure:

$$\begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

So it is legitimately a different solution.

$y_2(t) = t^{-2}$ so the general solution is

$$y(t) = c_1 t + c_2 t^{-2}$$

One can show in general that this procedure always works in forming a first order equation in v' .

The book shows this too.