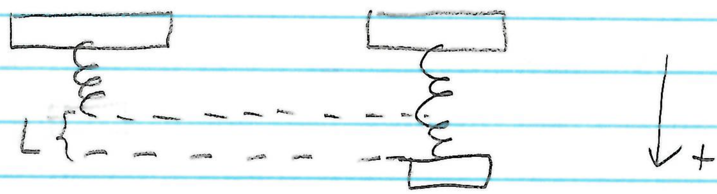
Lesson 19

Undamped Free Vibrations

Imagine a spring hanging from the ceiling. Now imagine a mass being attached to the spring and stretching it some.
(We assume ↓ is the positive direction)



By Hooke's Law, the spring force is proportional to the length L that the mass stretches the spring. At equilibrium, we have

$$F = ma$$

$$mg - kL = 0$$, where k is the spring constant.

When the mass is in motion, for undamped free vibrations, we assume the only forces acting on the mass are gravity and the spring force.

Let $u(t)$ be the displacement of the mass at time t.

$$F = ma$$

$$\text{gravity} - \text{spring force} = mu''(t)$$

$$mg - k(L+u) = mu''$$

(spring force always works opposite to velocity)

Get the diff eq.

$$mu'' + kL + ku - mg = 0$$

But $kL - mg = 0$

So for undamped, free vibrations, we have

$$\boxed{mu'' + ku = 0}$$

where u is the displacement of a mass with mass m on a spring with constant k.

Notice that m and k are positive constants, so the characteristic polynomial is

$$mr^2 + k$$

with roots $r = \pm\sqrt{\frac{-k}{m}}$, which are purely imaginary.

Hence, the general solution is

$$u(t) = c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

Analysis of Oscillation

If we have undamped, free vibrations, we like to analyze the amplitude, frequency, phase shift, and period of such vibrations. To do this, we want to write our solutions in the form

$$u(t) = R\cos(\omega_0 t - \delta)$$

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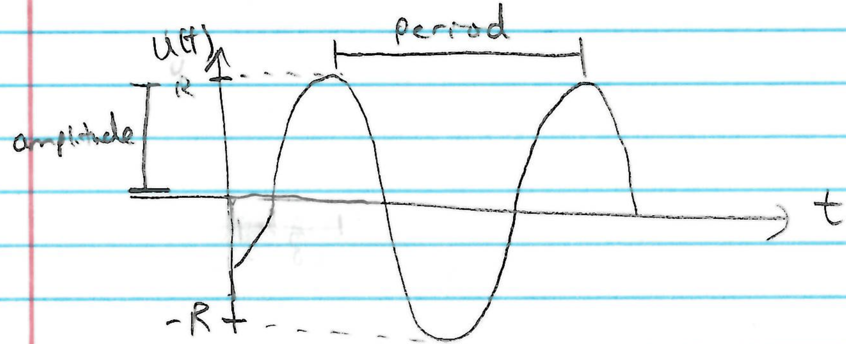

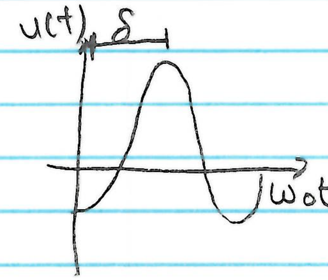

Part of $R\cos(\omega_0 t - \delta)$

R is the amplitude (how high the waves go)

$T = \frac{2\pi}{\omega_0}$ is the period (amount of time it takes to get from one part of the wave to the same part of the "next" wave)

ω_0 is the natural frequency (number of full waves times 2π that occur in a unit time interval)

δ is the phase shift (how far the wave is off from starting at $t=0$) in radians

So given $A\cos(\omega_0 t) + B\sin(\omega_0 t)$, want to convert to $R\cos(\omega_0 t - \delta)$

(Notice: $\omega_0 = \sqrt{\frac{k}{m}}$ in this scenario)

By the angle sum formulas,

$$R\cos(\omega_0 t - \delta) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t)$$

If $= A\cos(\omega_0 t) + B\sin(\omega_0 t)$, it follows that

$A = R\cos(\delta)$ and $B = R\sin(\delta)$

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To solve for R, notice

$$A^2 + B^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta) = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2$$

so $R = \sqrt{A^2 + B^2}$

To solve for δ, notice

$$\frac{B}{A} = \frac{R\sin(\delta)}{R\cos(\delta)} = \tan(\delta)$$

so $\delta = \tan^{-1}\left(\frac{B}{A}\right)$ (in the correct quadrant... may need to add π)
↳ analyze signs of B, A to get signs of $\sin(\delta), \cos(\delta)$.

Ex 1. Write $u = -\cos(3t) + \sqrt{3}\sin(3t)$ in $R\cos(\omega_0 t - \delta)$ form.

Known: $\omega_0 = 3$, $A = -1$, $B = \sqrt{3}$

$$R = \sqrt{A^2+B^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\delta = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3})$$

Now $B = R\sin(\delta) > 0$, and $A = R\cos(\delta) < 0$,
So δ must be in quadrant II.

S II	I A	
T III	IV C	

$\tan^{-1}(-\sqrt{3})$ is in quadrant IV

so $\delta = \tan^{-1}(-\sqrt{3}) + \pi$

$$\boxed{u = 2\cos(3t - \tan^{-1}(-\sqrt{3}) - \pi) \approx 2\cos(3t - 2.0944)}$$

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Ex 2. A mass of 100 g stretches a spring 3 cm. If the mass is set in motion from 1 cm below equilibrium position with a downward velocity of 40 cm/s, and there is no damping, determine the amplitude, period, natural frequency, and phase shift of the system.

Know: $mg - kL = 0 \Rightarrow (0.1\ kg)(9.8\ m/s^2) - k(0.03\ m) = 0$

$\Rightarrow k = \frac{0.98}{0.03} = \frac{98}{3}\ \frac{kg}{s^2}$

$mu'' + ku = 0, \quad u(0) = 0.01m, u'(0) = 0.04\ m/s$

$0.1u'' + \frac{98}{3}u = 0$

$0.1r^2 + \frac{98}{3} = 0$

$r^2 = -\frac{980}{3} \Rightarrow r = \pm i\sqrt{\frac{980}{3}}$

$u(t) = C_1 \cos\left(\sqrt{\frac{980}{3}}\, t\right) + C_2 \sin\left(\sqrt{\frac{980}{3}}\, t\right)$

$0.01 = u(0) = C_1$

$u'(t) = -0.01\sqrt{\frac{980}{3}} \sin\left(\sqrt{\frac{980}{3}}\, t\right) + C_2\sqrt{\frac{980}{3}} \cos\left(\sqrt{\frac{980}{3}}\, t\right)$

$0.04 = u'(0) = C_2\sqrt{\frac{980}{3}} \Rightarrow C_2 \approx 0.002213$

$u(t) \approx 0.01\cos\left(\sqrt{\frac{980}{3}}\, t\right) + 0.002213\sin\left(\sqrt{\frac{980}{3}}\, t\right)$

natural frequency $\omega_0 = \sqrt{\frac{980}{3}} \approx \boxed{18.0739 \text{ radians/sec}}$

$A = 0.01, \quad B = 0.00213$

$R = \sqrt{A^2 + B^2} \approx \boxed{0.0102\ m}$ amplitude

$T = \frac{2\pi}{\omega_0} \approx \boxed{0.3476 \text{ seconds}}$ period

$\delta = \tan^{-1}\left(\frac{B}{A}\right) \approx 0.2213$ (correct quadrant)

$\boxed{0.2213 \text{ radians}}$ phase shift