Lesson 2

Some Solutions and Classifications (1.2, 1.3)

Consider a diff eq of the form $\frac{dy}{dt} = ay - b$.
We can solve this using some clever techniques.
Notice:

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right)$$

Then: $$\frac{dy}{y-(b/a)} = a\,dt$$

Integrating both sides, we obtain:

$$\ln\left|y - \frac{b}{a}\right| = at + C$$

Exponentiate: $$\left|y - \frac{b}{a}\right| = e^{at+C} = e^{at} \cdot e^{C}$$ ← this is just a constant!

$$\left|y - \frac{b}{a}\right| = ce^{at}$$

$$y - \frac{b}{a} = \pm ce^{at}$$ just a constant!

$$y = ce^{at} + \frac{b}{a}$$

c is an arbitrary constant which results from integration. Whatever you plug in for c, it is a valid solution.

Ex 1: Solve $\frac{dy}{dt} = 2y - 3$.

$$\frac{dy}{dt} = 2\left(y - \frac{3}{2}\right)$$

$$\frac{dy}{y-(3/2)} = 2\,dt$$

$$\ln\left|y - \frac{3}{2}\right| = 2t + C$$

$$\left|y - \frac{3}{2}\right| = ce^{2t}$$

$$y = ce^{2t} + \frac{3}{2}$$

Or could just use the formula.

If we keep c as an arbitrary constant, y is said to be the <u>general solution</u>

If we choose a value for c, y is called an <u>integral curve</u>.

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Sometimes, we are looking for a particular solution. In order to find the particular solution, we are often given an <u>initial condition</u> $y(0) = y_0$.

A differential equation with an initial condition is called an <u>initial value problem (IVP)</u>.

<u>Ex 2</u>: Solve the IVP $\frac{dy}{dt} = 3y - 6$, $y(0) = 7$

General solution: $y = ce^{3t} + 2$

Since $y(0) = 7$, we get $7 = ce^{3(0)} + 2$

$7 = c + 2$

so $c = 5$

Thus, the solution is $y(t) = 5e^{3t} + 2$

<u>Ex 3</u>. Verify that $y(t) = 5e^{3t} + 2$ is a solution to $\frac{dy}{dt} = 3y - 6$.

First, $\frac{dy}{dt} = 15e^{3t}$

Now, $3y - 6 = 3(5e^{3t} + 2) - 6$

$= 15e^{3t} + 6 - 6 = 15e^{3t}$

Since $\frac{dy}{dt} = 3y - 6$, this is indeed a solution.

<u>Ex 4</u>. Verify that $y = e^{2t}$ is a solution to

$y''' - 2y'' + 2y' - 4y = 0$

$y' = 2e^{2t}$, $y'' = 4e^{2t}$, $y''' = 8e^{2t}$

$(8e^{2t}) - 2(4e^{2t}) + 2(2e^{2t}) - 4(e^{2t}) \stackrel{?}{=} 0$

$8e^{2t} - 8e^{2t} + 4e^{2t} - 4e^{2t} \stackrel{\checkmark}{=} 0$

We can also do some other clever techniques knowing what we know so far.

Ex 5. Suppose a bacteria population grows at a rate proportional to its current population, i.e., $\frac{dp}{dt} = rp$ where $p(t)$ is population and r is the rate. If the population triples in 2 hours, what is r?

$$\frac{dp}{dt} = rp$$
$$\frac{dp}{p} = r\,dt$$
$$\ln|p| = rt + C$$
$$p = ce^{rt}$$

$p(0) = c$, so $p(2) = 3c$

$$3c = ce^{2r}$$
$$3 = e^{2r}$$
$$\ln 3 = 2r$$
$$r = \frac{\ln 3}{2}.$$

Ex 6. For what value(s) of r is $y = e^{rt}$ a solution to $y'' - y' - 6y = 0$?

$y' = re^{rt}$, $y'' = r^2e^{rt}$

$$(r^2e^{rt}) - (re^{rt}) - 6(e^{rt}) = 0$$
$$e^{rt}(r^2 - r - 6) = 0$$
$$e^{rt}(r+2)(r-3) = 0$$
$$\boxed{r = -2 \text{ or } r = 3}$$

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Classifying differential equations:

If a differential equation has partial derivitives, then it is a partial diff eq. If it has only ordinary derivitives, then it is an ordinary diff eq.

Ex: $\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 0$ is ordinary

$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0$ is partial

The order of a differential equation is the order of the highest derivative that appears in the equation.

Ex: $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$ is order 2

$\frac{d^4y}{dt^2} + y = 0$ is order 4

$\left(\frac{d^3y}{dt^3}\right)^2 + y\frac{dy}{dt} = 1$ is order 3

An equation in n variables $x_1, \dots, x_n$ is linear if it is in the form

$a_1x_1 + \dots + a_nx_n = b$ where $a_1, \dots, a_n, b$ are not variables.

In a differential equation, we count y and its derivatives $(y', y'', \dots, y^{(n)})$ as the variables.

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A differential equation is linear if it is of the form

$$g_n(t)y^{(n)} + \dots + g_1(t)y' + g_0(t)y = f(t)$$

where $g_n(t), \dots, g_0(t), f(t)$ are functions of the independent variable t.

Ex: $t^2 \frac{d^2y}{dt^2} + 3\frac{dy}{dt} = \cos t$ is linear

$e^t \frac{d^2y}{dt} + \cos t \frac{dy}{dt} + y^2 = 1$ is nonlinear (because y is squared)

$y\frac{dy}{dt} + 3 = 0$ is nonlinear (because y is multiplied by $\frac{dy}{dt}$)

Classifying differential equations is important since some techniques of solving differential equations only apply to certain classes of differential equations.

Lesson 3 covers the integrating factor method, which applies to first order linear ODEs.