Lesson 20
Damped Free Vibrations

In this lesson, we assume there is a damping force acting on the mass proportionally to velocity in the direction opposite to velocity.

In other words, the damping force is given by $-\gamma u'(t)$, for some damping coefficient γ.

Like in lesson 19, we have the following diff eq.

$$\underbrace{mu''}_{ma} = \underbrace{-\gamma u'}_{\text{damping force}} + \underbrace{mg}_{\text{gravity}} - \underbrace{k(L+u)}_{\text{spring force}}$$

$$mu'' + \gamma u' - ku + \underbrace{mg - kL}_{=0} = 0$$

$$\boxed{mu'' + \gamma u' - ku = 0}$$

Ex 1. A 500 gram mass stretches a spring 5 cm. Suppose the mass is stretched an additional 2 cm downward from its equilibrium position and released. If a damper with coefficient 20 $\text{dyn}\cdot\text{s}/\text{cm}$ acts on the mass, set up an IVP modeling this scenario.

Make sure units are consistent!

$1 \text{ dyne} = 1 \ \frac{\text{g}\cdot\text{cm}}{\text{s}^2}$

$g = 9.8 \text{ m/s}^2 = \frac{9.8 \text{ m}}{\text{s}^2} \cdot \frac{100 \text{cm}}{1 \text{m}} = 980 \text{ cm/s}^2$

$mg - kL = 0$

$(500\text{g})(980 \text{ cm/s}^2) - k(5 \text{ cm}) = 0$

$k = 98{,}000 \text{ g/s}^2$

$\gamma = 20 \frac{\text{dyn}\cdot\text{s}}{\text{cm}}$, $m = 500\text{ g}$

$$500u'' + 20u' + 98{,}000u = 0$$

Initially stretched downward 2 cm (positive direction)
so $u(0) = 2$
"Released" implies initial velocity is 0, so $u'(0) = 0$.

$$\boxed{500u'' + 20u' + 98{,}000u = 0, \quad u(0) = 2, \quad u'(0) = 0}$$

($u(t)$ is measured in cm, t in seconds)

<u>Analyzing Damped Free Vibrations</u>

With a nonzero damping force, roots of characteristic polynomial are of the form

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Since $\gamma \neq 0$, roots cannot be purely imaginary, so we do <u>not</u> have steady oscillation.

The system is said to be:
- <u>critically damped</u> if $\gamma^2 = 4mk$ (repeated real root)
- <u>overdamped</u> if $\gamma^2 > 4mk$ (distinct real roots)
- <u>underdamped</u> if $\gamma^2 < 4mk$ (complex roots)

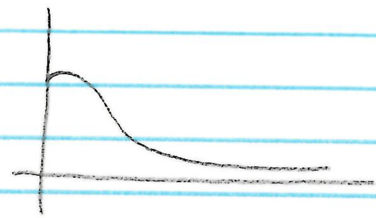
<u>Critically Damped Case</u>

roots are $\frac{-\gamma}{2m}$ (repeated)

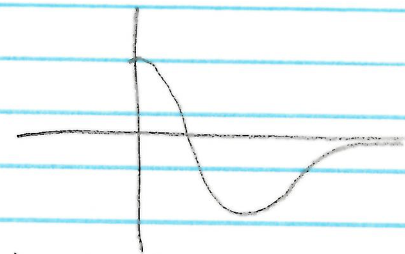
so $u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$

Lesson 20

graphs look like



or



mass tends to "slide" back to equilibrium position as fast as possible without bouncing, or bouncing only once.

"Ideal" amount of damping

Critical, since if you change the damping at all, no matter how slightly, behavior will change.

Ex 2. A mass of 700 grams stretches a spring 8 cm. Find the value of γ for which the system is critically damped.

write $m = 0.7$ kg, $L = 0.08$ m

$$mg - kL = 0$$
$$(0.7)(9.8) - k(0.08) = 0$$
$$k = 85.75 \ \frac{kg}{s^2}$$

$$0.7u'' + \gamma u' + 85.75u = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(0.7)(85.75)}}{2(0.7)}$$

want $\gamma^2 = 4(0.7)(85.75)$
$= 240.1$

so $\gamma \approx 15.4952$

$\gamma u'$ is a force, measured in Newtons (how we set it up).
u' measured in m/s.

Lesson 20

$$(\text{units of } \gamma)\cdot \frac{m}{s} = N$$

$$\text{units of } \gamma = \frac{N\cdot s}{m}$$

$$\boxed{\gamma \approx 15.4952 \ \frac{N\cdot s}{m}}$$

Overdamped Case

roots are distinct and real, $-\alpha \neq -\beta$

but must both be negative

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \qquad \left(\begin{array}{l} \gamma^2 - 4mk < \gamma^2 \\ \text{so } \sqrt{\gamma^2-4mk} < \gamma \end{array} \right)$$

so $u(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t}$

graphs look similar to critically damped case, but approach equilibrium more slowly. Face "too much" damping to slide back easily.

Underdamped Case

complex conjugate roots, $-\lambda \pm \mu i$

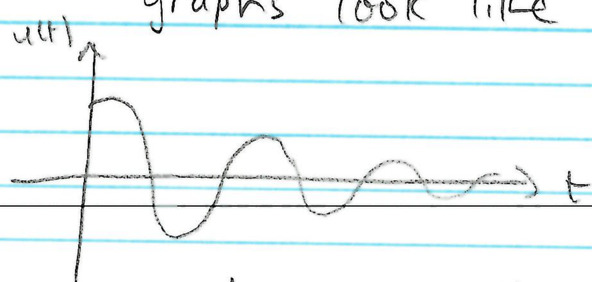
negative real part

$$\frac{-\gamma \pm \sqrt{\gamma^2-4mk}}{2m}$$

real part

so $u(t) = c_1 e^{-\lambda t} \cos \mu t + c_2 e^{-\lambda t} \sin \mu t$

graphs look like



decaying oscillations

Slows down, but is not damped enough to avoid some oscillations.

Even though the oscillations don't have a "period" or "frequency" (they are not steady), they have something close.

$$A e^{-\lambda t} \cos \mu t + B e^{-\lambda t} \sin \mu t$$
$$= e^{-\lambda t} (A \cos \mu t + B \sin \mu t)$$
$$= e^{-\lambda t} R \cos(\mu t - \delta)$$

μ is the <u>quasi-frequency</u> of the oscillations

$T_d = \dfrac{2\pi}{\mu}$ is the <u>quasi-period</u>
(amount of time between successive "peaks" or "valleys")

<u>Ex 3</u>. A mass of 15 g stretches a spring 3 cm. Suppose it is damped with coefficient 500 $\frac{\text{dyn}\cdot\text{s}}{\text{cm}}$. Suppose the mass is pulled down an additional cm and then released. Find the quasi-frequency and quasi-period. Compare it to the natural frequency and period of the corresponding undamped system. Plot both curves.

$$mg - kL = 0 \qquad g = \frac{9.8\,m}{s^2} \cdot \frac{100\,cm}{m} = \frac{980\,cm}{s^2}$$
$$(15\,g)(980\,\tfrac{cm}{s^2}) - k(3\,cm) = 0$$
$$k = 4900\ g/s^2$$

$$15u'' + 500u' + 4900u = 0$$

$$r = \frac{-500 \pm \sqrt{(500)^2 - 4(15)(4900)}}{2(15)} = -\frac{50}{3} \pm i\frac{2}{3}\sqrt{110}$$

$$u(t) = c_1 e^{-50t/3}\cos\left(\tfrac{2}{3}\sqrt{110}\,t\right) + c_2 e^{-50t/3}\sin\left(\tfrac{2}{3}\sqrt{110}\,t\right)$$

using $u(0)=1$ and $u'(0)=0$, get

$$u(t) = e^{-50t/3}\cos\left(\tfrac{2}{3}\sqrt{110}\,t\right) + \frac{25}{\sqrt{110}} e^{-50t/3}\sin\left(\tfrac{2}{3}\sqrt{110}\,t\right)$$

quasi-frequency $= \frac{2}{3}\sqrt{110} \approx 6.992$

quasi-period $= \dfrac{2\pi}{\left(\frac{2}{3}\sqrt{110}\right)} \approx 0.899$

Corresponding undamped system:

$$15u'' + 4900u = 0$$

$$r = \frac{\pm\sqrt{-4(15)(4900)}}{2(15)} = \pm i \cdot \frac{14}{3}\sqrt{15}$$

$$u(t) = c_1\cos\left(\tfrac{14}{3}\sqrt{15}\,t\right) + c_2\sin\left(\tfrac{14}{3}\sqrt{15}\,t\right)$$

using $u(0)=1$ and $u'(0)=0$, get

$$u(t) = \cos\left(\tfrac{14}{3}\sqrt{15}\,t\right)$$

natural frequency $\omega_0 = \frac{14}{3}\sqrt{15} \approx 18.074$

period $= \dfrac{2\pi}{\left(\frac{14}{3}\sqrt{15}\right)} \approx 0.348$

damping decreases frequency and increases period.

Sketch graphs on Desmos