Lesson 23

Method of Undetermined Coefficients (4.3)

The Method of Undetermined Coefficients is very similar for nth order equations compared to 2nd order.

1. Find the complementary solution
2. Analyze the terms of $g(t)$ and make initial guesses for terms of $Y(t)$.
3. If any term of the initial guess is a term in the complementary solution, multiply by a high enough power of t (lowest power of t which makes all terms not part of the complementary solution).
4. Find $Y'(t)$, $Y''(t)$, ..., $Y^{(n)}(t)$ and plug into the diff eq, and solve for each coefficient.
5. General solution: $y_c(t) + Y(t)$

Ex 1. Find the general solution.

$$y''' - y'' - y' + y = 2e^{-t} + 3$$

$$r^3 - r^2 - r + 1 = 0$$

$$r^2(r-1) - (r-1) = 0$$

$$(r^2-1)(r-1) = 0$$

$$(r+1)(r-1)^2 = 0 \qquad r_1 = 1,\ r_2 = 1,\ r_3 = -1$$

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}$$

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$$g(t) = 2e^{-t} + 3$$

terms:	$2e^{-t}$	3
guess 1:	Ae^{-t}	B
in y_c?:	yes	no
guess 2:	Ate^{-t}	same as guess 1

$$Y(t) = Ate^{-t} + B$$

$$Y'(t) = -Ate^{-t} + Ae^{-t} = (-At+A)e^{-t}$$
$$Y''(t) = (At-A)e^{-t} + (-A)e^{-t} = (At-2A)e^{-t}$$
$$Y'''(t) = (-At+2A)e^{-t} + (A)e^{-t} = (-At+3A)e^{-t}$$

Plug in to
$$y''' - y'' - y' + y = 2e^{-t} + 3$$

$$(-At+3A)e^{-t} - (At-2A)e^{-t} - (-At+A)e^{-t} + Ate^{-t} + B = 2e^{-t} + 3$$

$$(\underline{-At} + 3A \underline{-At} + 2A \underline{+At} - A \underline{+At})e^{-t} + B = 2e^{-t} + 3$$
$$4Ae^{-t} + B = 2e^{-t} + 3$$

$$4A = 2, \text{ so } A = \tfrac{1}{2}; \quad B = 3$$

$$Y(t) = \tfrac{1}{2}te^{-t} + 3$$

$$y(t) = y_c(t) + Y(t)$$

$$\boxed{y(t) = c_1e^{t} + c_2te^{t} + c_3e^{-t} + \tfrac{1}{2}te^{-t} + 3}$$

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Ex 2. Find an appropriate form for $Y(t)$ for
$$y^{(5)} - 15y^{(4)} + 90y''' - 270y'' + 405y' - 243y = 2e^{3t}$$

Characteristic polynomial factors as $(r-3)^5$.
$$y_c(t) = c_1 e^{3t} + c_2 te^{3t} + c_3 t^2 e^{3t} + c_4 t^3 e^{3t} + c_5 t^4 e^{3t}$$

$$g(t) = 2e^{3t}$$

term: $2e^{3t}$
guess 1: Ae^{3t}
in y_c?: yes
guess 2: $At^5 e^{3t}$ ← here, we multiply by t^5 since this is the lowest power of t which is not in y_c.

$$\boxed{Y(t) = At^5 e^{3t}}$$

Ex 3. Find an appropriate form for $Y(t)$ for
$$y^{(5)} = 3t^5 e^{3t}$$

Characteristic polynomial is r^5
$$y_c(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4$$

$$g(t) = 3t^5 e^{3t}$$

term: $3t^5 e^{3t}$
guess 1: $(A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0)e^{3t}$
in y_c?: no! no terms are in y_c.
guess 2: same as guess 1.

$$\boxed{Y(t) = (A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0)e^{3t}}$$

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Understanding the difference between examples 2 and 3 is crucial for mastering the Method of Undetermined Coefficients.

In example 2, our initial guess was a single term but was in y_c, so we had to multiply it by a sufficiently high power of t to get it out of y_c.

In example 3, our initial guess reflected that the original term had a 5^{th} degree polynomial times an exponential function. No part of our initial guess was in y_c, so the initial guess worked.

<u>Ex 4.</u> Suppose $y_c(t) = c_1 e^{4t} + c_2 t e^{4t} + c_3 t^2 e^{4t}$ and $g(t) = 3t^3 e^{4t}$. Find an appropriate form for $Y(t)$.

term: $3t^3 e^{4t}$

guess 1: $(At^3 + Bt^2 + Ct + D)e^{4t}$

in y_c?: yes, specifically, last 3 terms here

guess 2: $t^3(At^3 + Bt^2 + Ct + D)e^{4t}$

$= (At^6 + Bt^5 + Ct^4 + Dt^3)e^{4t}$

Notice that even though At^3e^{4t} is not in y_c, part of guess 1 is, so we need to multiply by a power of t. The lowest power which gets <u>every</u> part of guess 1 out of y_c is t^3.

$$\boxed{Y(t) = (At^6 + Bt^5 + Ct^4 + Dt^3)e^{4t}}$$

Advanced Techniques for "Experts".

If you do not feel confident that you have mastered finding a suitable form for $Y(t)$, you can ignore this. These are time-saving tricks.

Notice:

All even derivatives of $A\cos t$ are $C\cos t$
All even derivatives of $A\sin t$ are $C\sin t$
All odd derivatives of $A\cos t$ are $C\sin t$
All odd derivatives of $A\sin t$ are $C\cos t$

Ex. For $y^{(4)} - 5y'' - 36y = 4\cos t$,
$y_c(t) = C_1 e^{3t} + C_2 e^{-3t} + C_3\cos(2t) + C_4\sin(4t)$.

Our appropriate form here would normally be
$Y(t) = A\cos t + B\sin t$.
Notice that only even derivatives appear in the diff eq, so $A\cos t$ only contributes terms of the form $C\cos t$ when plugged in, and $B\sin t$ only contributes terms of the form $C\sin t$ when plugged in.
Since $g(t)$ has no terms containing $\sin t$, in this instance, we can get rid of $B\sin t$.
$Y(t) = A\cos t$ is sufficient.

Warning: $At\cos t$, $Bt\sin t$ will get both trig functions in all derivatives, so this trick does not work when you have to do the product rule.

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Ex. For $y''' - 4y' = \cos(3t) + t^2$,

$$y_c(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}$$

Our appropriate form here would normally be

$$Y(t) = A\cos(3t) + B\sin(3t) + t(C_2 t^2 + C_1 t + C_0)$$
$$= A\cos(3t) + B\sin(3t) + C_2 t^3 + C_1 t^2 + C_0 t$$

Since only odd derivatives appear here, $A\cos(3t)$ only contributes terms of the form $C\sin(3t)$ and $B\sin(3t)$ only contributes terms of the form $C\cos(3t)$. Since $g(t)$ has no terms containing $\sin(3t)$ and since $Y(t)$ does not require the product rule, we can eliminate $A\cos(3t)$, leaving us with

$$Y(t) = A\sin(3t) + B_2 t^3 + B_1 t^2 + B_0 t$$

Also, since only odd derivatives appear, for the polynomial part, we will be subtracting only odd numbers from exponents of t.

odd − odd = even

even − odd = odd

Since $g(t)$ only contains even powers of t, want odd − odd case (resulting in even), so even powers of t in $Y(t)$ contribute nothing, so we can eliminate $B_1 t^2$, leaving us with

$$Y(t) = A\sin(3t) + Bt^3 + Ct$$

(A constant is a term containing an even power of t, namely t^0. Keep this in mind when using this trick.)