Lesson 25

Laplace Transform and IVPs (6.2)

In lesson 20, we looked at the Laplace Transform of a function. A natural question to ask is the following: Can we invert the Laplace transform? It turns out "no" but for our purposes, "yes"!

Fact: If $F(s) = \mathcal{L}\{y\}$ for some function y, then there is exactly one continuous function ϕ with $F(s) = \mathcal{L}\{\phi\} = \mathcal{L}\{y\}$.
i.e., $\phi = \mathcal{L}^{-1}\{F(s)\}$.

Fact. Both the Laplace transform $\mathcal{L}$ and the inverse Laplace transform $\mathcal{L}^{-1}$ are linear operators. i.e., $\mathcal{L}\{c_1 y_1 + c_2 y_2\} = c_1 \mathcal{L}\{y_1\} + c_2 \mathcal{L}\{y_2\}$, and $\mathcal{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1 \mathcal{L}^{-1}\{F(s)\} + c_2 \mathcal{L}^{-1}\{G(s)\}$.

You can check the linearity of $\mathcal{L}$ by the definition. There is technically a formula for the inverse Laplace transform, but it involves heavy duty complex analysis. As such, we use our knowledge of Laplace transforms to compute the inverse Laplace Transform.

A table of Laplace Transforms of common continuous functions can be found in the back cover of your textbook.
You may use this from here on out.

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Ex 1. Compute $\mathcal{L}^{-1}\{F(s)\}$ when

(a) $F(s) = \dfrac{3s}{s^2-s-6}$ Nothing on the table looks like this, so use partial fraction decomposition!

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$A(s+2) + B(s-3) = 3s$

If $s=3$, $5A = 9$, so $A = \frac{9}{5}$

If $s=-2$, $-5B = -6$, so $B = \frac{6}{5}$

Thus, $F(s) = \frac{9}{5}\left(\frac{1}{s-3}\right) + \frac{6}{5}\left(\frac{1}{s+2}\right)$

From line 2 on the Laplace Transform Table,

$$\mathcal{L}^{-1}\{F(s)\} = \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$
$$= \boxed{\frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}}$$

(b) $F(s) = \dfrac{1-2s}{s^2+4s+5}$ Nothing on the table looks like this, but can't use partial fractions since s^2+4s+5 is irreducible over the reals.

As such, we complete the square

$$s^2+4s+\underline{4}+5-\underline{4}$$
$$= (s^2+4s+4)+1$$
$$= (s+2)^2+1$$

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$$F(s) = \frac{1-2s}{(s+2)^2+1}$$ looks like lines 9 + 10 but not exactly.

$$= \frac{-2(s+2)+5}{(s^2+2)+1} = -2\left(\frac{s+2}{(s+2)^2+1}\right) + 5\left(\frac{1}{(s+2)^2+1}\right)$$

Thus, by lines 9 + 10

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = -2e^{-2t}\cos t + 5e^{-2t}\sin t}$$

Why do we care about the Laplace Transform? The following Theorem and Corollary lead to the answer of this question:

<u>Theorem 6.2.1.</u> If f is continuous and f' is piecewise continuous on an appropriate interval and $\mathcal{L}\{f(t)\}$ exists, then $\mathcal{L}\{f'(t)\}$ exists and $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

The proof is given in the textbook.

<u>Corollary 6.2.2.</u> If $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on an appropriate interval and $\mathcal{L}\{f\}$ exists, then $\mathcal{L}\{f^{(n)}\}$ exists and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots$$

$$- s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

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So if we are given an IVP with $t_0 = 0$ for the initial condition, we can convert the IVP to an algebraic equation using the Laplace transform, perform algebraic manipulations, then transform back using $\mathcal{L}^{-1}$ to get the solution.

Ex 2. Use the Laplace Transform to solve the IVP.

$$y'' - 2y' + 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 1$$

Apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

(where $Y(s) = \mathcal{L}\{y\}$)

$$s^2 Y(s) - 0 - 1 - 2sY(s) + 0 + 2Y(s) = \frac{1}{s+1}$$

$$s^2 Y(s) - 2s Y(s) + 2Y(s) = \frac{1}{s+1} + 1$$

$$(s^2 - 2s + 2) Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}$$

Since $Y(s) = \mathcal{L}\{y\}$, $y = \mathcal{L}^{-1}\{Y(s)\}$

$s^2 - 2s + 2$ is irreducible over the reals

$$s^2 - 2s + 2 = s^2 - 2s + \underline{1} + 2 - \underline{1}$$

$$= (s-1)^2 + 1$$

$$Y(s) = \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 2} + \frac{1}{(s-1)^2 + 1}$$

$$A(s^2 - 2s + 2) + (Bs + C)(s+1) = 1$$

$$As^2 - 2As + 2A + Bs^2 + Bs + Cs + C = 1$$

$$(A+B)s^2 + (-2A + B + C)s + (2A + C) = 1$$

$$\begin{cases} A + B = 0 \\ -2A + B + C = 0 \\ 2A + C = 1 \end{cases} \Rightarrow A = \frac{1}{5}, \; B = -\frac{1}{5}, \; C = \frac{3}{5}$$

$$Y(s) = \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{-\frac{1}{5}s+\frac{3}{5}}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

$$= \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{-\frac{1}{5}s+\frac{8}{5}}{(s-1)^2+1}$$

$$= \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{-\frac{1}{5}(s-1)+\frac{7}{5}}{(s-1)^2+1}$$

$$= \frac{1}{5}\underbrace{\left(\frac{1}{s+1}\right)}_{\text{line 2}} - \frac{1}{5}\underbrace{\left(\frac{s-1}{(s-1)^2+1}\right)}_{\text{line 10}} + \frac{7}{5}\underbrace{\left(\frac{1}{(s-1)^2+1}\right)}_{\text{line 9}}$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t}$$

Ex 3. Find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution to the IVP.

$$y'' + y = \underbrace{\begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}}_{g(t)} \quad ; \; y(0) = 0, \; y'(0) = 0$$

$g(t)$ is not on our table, so we have to compute $\mathcal{L}\{g(t)\}$

$$= \int_0^\infty e^{-st} g(t)\,dt = \int_0^1 e^{-st}\,t\,dt + \underbrace{\int_1^\infty 0\,dt}_{=0}$$

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{e^{-st}}{s}$

$$\mathcal{L}\{g(t)\} = -\frac{t e^{-st}}{s}\Big|_{t=0}^{t=1} - \int_0^1 -\frac{e^{-st}}{s}\,dt = \frac{-e^{-s}}{s} - \left[\frac{e^{-st}}{s^2}\right]_{t=0}^{t=1}$$

$$= \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

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$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{(s+1)e^{-s}}{s^2}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2} - \frac{(s+1)e^{-s}}{s^2}$$

$$s^2 Y(s) - 0 - 0 + Y(s) = \frac{1}{s^2} - \frac{(s+1)e^{-s}}{s^2}$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2} - \frac{(s+1)e^{-s}}{s^2}$$

$$\boxed{Y(s) = \frac{1}{s^2(s^2+1)} - \frac{(s+1)e^{-s}}{s^2(s^2+1)}}$$