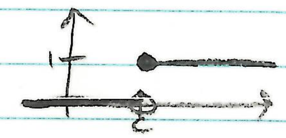
Lesson 26

Step (Heaviside) Functions (6.3)

The real power of the Laplace Transform comes from being able to handle discontinuous forcing functions. Before, the techniques we worked with did not allow the nonhomogeneous part of an ODE to have discontinuities. To solve diff eqs with discontinuous forcing functions, we look at step functions.

The <u>unit step function</u> or <u>Heaviside function</u> with a jump at c is defined as

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



<u>Ex 1</u>. Sketch the function and describe it as a piecewise defined function

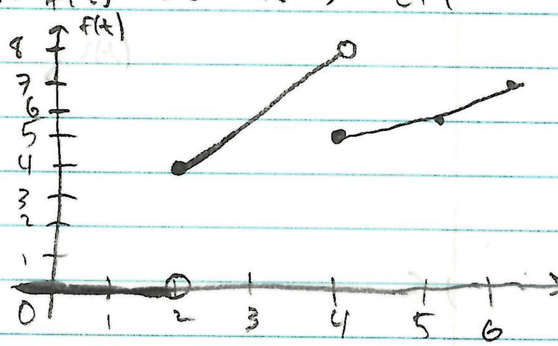
$$f(t) = 2t\,u_2(t) - (t-1)\,u_4(t)$$

When $t < 2$, $u_2(t) = u_4(t) = 0$, so $f(t) = 0$

$2 \leq t < 4$, $u_2(t) = 1$, so $f(t) = 2t$

$t \geq 4$, $u_4(t) = 1$, so $f(t) = 2t - (t-1) = t+1$

$$f(t) = \begin{cases} 0, & t < 2 \\ 2t, & 2 \leq t < 4 \\ t+1, & t \geq 4 \end{cases}$$



Lesson 26

<u>Ex 2</u>. Write $f(t)$ in terms of step functions.

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t-2, & 2 \le t < 3 \\ 0; & t \ge 3 \end{cases}$$

Changes occur at $t = 0, 1, 2, 3$: $f(t) = u_0(t)\cdot\alpha(t) + u_1(t)\cdot\beta(t) + u_2(t)\cdot\gamma(t) + u_3(t)\cdot\delta(t)$

First part $f(t) = u_0(t)\cdot t$ + stuff

For $1 \le t < 2$, want $t-1 = t + g(t)u_1(t)$

Clearly, $g(t) = -1$

so $f(t) = u_0(t)t - u_1(t)$ + stuff

For $2 \le t < 3$, want $t-2 = t-1 + g(t)u_2(t)$

Need $g(t) = -1$ again

so $f(t) = tu_0(t) - u_1(t) - u_2(t)$ + stuff

For $t \ge 3$, want $0 = t-2 + g(t)u_3(t)$

Need $g(t) = -t+2$

so $\boxed{f(t) = tu_0(t) - u_1(t) - u_2(t) - (t-2)u_3(t)}$

Now, how does the Laplace Transform interact with step functions?

<u>Thm 6.3.1</u> If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s).$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}.$$

<u>Proof</u>. $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$

$= \int_c^\infty e^{-st}f(t-c)\,dt$ (since $u_c(t) = 0$ for $t < c$)

Let $\xi = t-c$. Then $d\xi = dt$, and $t = \xi + c$

$= \int_0^\infty e^{-(\xi+c)s}f(\xi)\,d\xi$

$= \int_0^\infty e^{-cs}e^{-\xi s}f(\xi)\,d\xi$

$= e^{-cs}\int_0^\infty e^{-\xi s}f(\xi)\,d\xi = e^{-cs}\mathcal{L}\{f(\xi)\}$

Notice that we have $f(t-c)$. This is important!

Ex 3. Compute $\mathcal{L}\{f(t)\}$ where

$$f(t) = u_1(t) + t u_2(t) + (t^2+1) u_4(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{t u_2(t)\} + \mathcal{L}\{(t^2+1) u_4(t)\}$$

$t = g(t-2)$ for some function $g(t)$.

Plug in $t+2$ for t. Then $t+2 = g(t)$

$t^2+1 = h(t-4)$ for some function $h(t)$.

Plug in $t+4$ for t. Then $t^2+8t+17 = (t+4)^2+1 = h(t)$

So $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} + e^{-2s} G(s) + e^{-4s} H(s)$

where $G(s) = \mathcal{L}\{g(t)\}$, $H(s) = \mathcal{L}\{h(t)\}$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t+2\} = \mathcal{L}\{t\} + \mathcal{L}\{2\} = \frac{1}{s^2} + \frac{2}{s} = \frac{1+2s}{s^2}$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{t^2\} + 8\mathcal{L}\{t\} + \mathcal{L}\{17\} = \frac{2}{s^3} + \frac{8}{s^2} + \frac{17}{s} = \frac{2+8s+17s^2}{s^3}$$

$$\text{So } \mathcal{L}\{f(t)\} = \frac{e^{-s}s^2}{s^3} + \frac{e^{-2s}(1+2s)s}{s^3} + \frac{e^{-4s}(2+8s+17s^2)}{s^3}$$

$$= \left[e^{-s}s^2 + e^{-2s}(1+2s) + e^{-4s}(2+8s+17s^2)\right] / s^3$$

Ex 4. Compute $\mathcal{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{e^{-2s}}{s^2+s-2}$$

$$F(s) = e^{-2s}\left(\frac{1}{s^2+s-2}\right) = e^{-2s}\left(\frac{1}{(s+2)(s-1)}\right)$$

$$\frac{1}{(s+2)(s-1)} = \frac{a}{s+2} + \frac{b}{s-1} \Rightarrow a(s-1) + b(s+2) = 1$$

$s=1 \Rightarrow 3b = 1 \Rightarrow b = \frac{1}{3}$, $s=-2 \Rightarrow -3a = 1 \Rightarrow a = -\frac{1}{3}$

$$F(s) = -\frac{1}{3} e^{-2s}\left(\frac{1}{s+2}\right) + \frac{1}{3} e^{-2s}\left(\frac{1}{s-1}\right)$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{3}\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3} u_2(t)\, g(t-2) + \frac{1}{3} u_2(t)\, h(t-2)$$

where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$, $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$

so $g(t-2) = e^{-2(t-2)} = e^{-2t+4}$, $h(t-2) = e^{(t-2)} = e^{t-2}$

$$\mathcal{L}^{-1}\{F(s)\} = \left(-\frac{1}{3}e^{-2t+4} + \frac{1}{3}e^{t-2}\right) u_2(t)$$

Lesson 26

Thm 6.3.2. If $F(s) = \mathcal{L}\{f(t)\}$ for $s > a \geq 0$ and if c is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof. $\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt$

$$= \int_0^\infty e^{-(s-c)t} f(t)\, dt$$

$$= F(s-c)$$

Ex 5. Compute $\mathcal{L}\{f(t)\}$ where $f(t) = e^{3t} \sin(t)$

$\mathcal{L}\{f(t)\} = G(s-3)$ where $G(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

Then $G(s-3) = \frac{1}{(s-3)^2+1} = \frac{1}{s^2-6s+10}$

so $\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s^2-6s+10}}$

Ex 6. Compute $\mathcal{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{3s+6}{s^2+4s+8}$$

Notice, $F(s) = \frac{3(s+2)}{s^2+4s+\underline{4}+8-\underline{4}} = \frac{3(s+2)}{(s+2)^2+4} = 3G(s+2)$

where $G(s) = \frac{s}{s^2+4}$

$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\{G(s+2)\} = 3e^{2t} g(t)$

where $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} = \cos 2t$

so $\boxed{\mathcal{L}^{-1}\{F(s)\} = 3e^{2t}\cos 2t}$