Lesson 28

Impulse Functions (6.5)

Sometimes we want to deal with a forcing function $g(t)$ which is very large in magnitude for a very small amount of time, say $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and $g(t) = 0$ elsewhere.
We can describe the impulse of this function,

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt = \int_{-\infty}^{\infty} g(t)\,dt$$

Let's assume $t_0 = 0$, and that

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{elsewhere} \end{cases}$$

where τ is a small positive constant.

$$I(\tau) = \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt = \left[\frac{t}{2\tau}\right]_{t=-\tau}^{t=\tau} = \frac{\tau}{2\tau} - \frac{-\tau}{2\tau} = 1$$

so the impulse of this function is 1, regardless of the value of τ.

We now define a function $\delta(t) = \lim_{\tau \to 0^+} d_\tau(t)$.

Notice: $\lim_{\tau\to 0^+} I(\tau) = \lim_{\tau \to 0^+} \int_{-\tau}^{\tau} d_\tau(t)\,dt = \lim_{\tau\to 0^+} 1 = 1$,
so we define $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$.

So we obtain the <u>unit impulse function</u> or <u>Dirac delta function</u> $\delta(t)$ which is defined by the properties:

$$\delta(t) = 0 \text{ for all } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Lesson 28

So if we have a situation which calls for an impulse of value A at time t_0, we can represent that by $A\delta(t-t_0)$, which has the properties

$A\delta(t-t_0) = 0$ for all $t \neq t_0$

$\int_{-\infty}^{\infty} A\,\delta(t-t_0)\,dt = A\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = A$

To work with such functions in the context of differential equations, we need to define $\mathcal{L}\{\delta(t-t_0)\}$. Like with the impulse of δ, we define:

$$\mathcal{L}\{\delta(t-t_0)\} := \lim_{\tau\to 0^+} \mathcal{L}\{d_\tau(t-t_0)\}$$

$$\mathcal{L}\{d_\tau(t-t_0)\} = \int_0^{\infty} e^{-st} d_\tau(t-t_0)\,dt$$

$$= \int_{t_0-\tau}^{t_0+\tau} e^{-st} \frac{1}{2\tau}\,dt$$

$$= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt$$

$$= \frac{1}{2s\tau} e^{-st_0}\left(e^{s\tau} - e^{-s\tau}\right)$$

$$= \frac{\sinh s\tau}{s\tau} e^{-st_0}$$

Thus, $\mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau\to 0^+}\left(\frac{\sinh s\tau}{s\tau}\right)e^{-st_0}$

$$\overset{L'H}{=} \lim_{\tau\to 0^+} \frac{s\cosh s\tau}{s} e^{-st_0}$$

$$= \frac{s}{s} e^{-st_0}$$

$$= e^{-st_0}$$

So $\boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}}$

Lesson 28

Ex 1. Solve the IVP.

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi); \quad y(0)=0,\ y'(0)=0$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\} - \mathcal{L}\{\delta(t-2\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$(s^2+4)Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$

$$= u_\pi(t) f(t-\pi) - u_{2\pi}(t) f(t-2\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$

$$f(t-\pi) = \tfrac{1}{2}\sin[2(t-\pi)]$$

$$f(t-2\pi) = \tfrac{1}{2}\sin[2(t-2\pi)]$$

$$\boxed{y(t) = u_\pi(t)\left[\tfrac{1}{2}\sin[2(t-\pi)]\right] - u_{2\pi}(t)\left[\tfrac{1}{2}\sin[2(t-2\pi)]\right]}$$

Ex 2. Solve the IVP.

$$y'' + 2y' + 3y = \sin t + \delta(t-3\pi)\ ;\ y(0)=0,\ y'(0)=0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\delta(t-3\pi)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 3Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$$

$$(s^2+2s+3)Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + \frac{e^{-3\pi s}}{(s^2+2s+3)}$$

$$\frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3}$$

$$1 = (As+B)(s^2+2s+3) + (Cs+D)(s^2+1)$$

$$1 = As^3 + Bs^2 + 2As^2 + 2Bs + 3As + 3B + Cs^3 + Ds^2 + Cs + D$$

$$\begin{cases} A + C = 0 \\ 2A + B + D = 0 \\ 3A + 2B + C = 0 \\ 3B + D = 1 \end{cases} \Rightarrow A = -\frac{1}{4},\ B = C = D = \frac{1}{4}$$

Also $s^2 + 2s + 3 = s^2 + 2s + \underline{1} + 3 - \underline{1} = (s+1)^2 + 2$

$$Y(s) = \frac{1}{4}\Big(\underbrace{\frac{-s+1}{s^2+1}}_{\frac{-s}{s^2+1} + \frac{1}{s^2+1}} + \frac{s+1}{(s+1)^2+2}\Big) + e^{-3\pi s}\Big(\underbrace{\frac{1}{(s+1)^2+2}}_{\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}}\Big),$$

$$y(t) = \frac{1}{4}\left(-\cos t + \sin t + e^{-t}\cos(\sqrt{2}t)\right) + \frac{1}{\sqrt{2}}u_{3\pi}(t)\,f(t-3\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+2}\right\} = e^{-t}\sin(\sqrt{2}t)$

$$\boxed{y(t) = \frac{1}{4}\left(\sin t - \cos t + e^{-t}\cos(\sqrt{2}t)\right) + \frac{1}{\sqrt{2}}u_{3\pi}\left[e^{-(t-3\pi)}\sin[\sqrt{2}(t-3\pi)]\right]}$$