Lesson 3

Integrating Factors (2.1)

In this lesson we focus on first order linear ODEs. Recall that such an ODE is of the form

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

(If $\frac{dy}{dt}$ has a coefficient function, we can divide by it to put it in this form)

Gottfried Leibniz noticed that the LHS of this equation looks very similar to the result of the product rule on some function $\mu(t)\,y$. (Recall: $\frac{d}{dt}[\mu(t)y] = \mu(t)\cdot\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$)
We want to find a function $\mu(t)$ so that $\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y]$.

$\mu(t)$ is called the integrating factor.

So we want $\mu(t)\frac{dy}{dt} + \mu(t)p(t)\,y = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$

Subtracting $\mu(t)\frac{dy}{dt}$ from both sides and dividing both sides by y, we get the diff eq

$$\frac{d\mu(t)}{dt} = p(t)\,\mu(t)$$

$$\frac{d\mu(t)}{\mu(t)} = p(t)\,dt$$

$$\ln|\mu(t)| = \int p(t)\,dt$$

so $\mu(t) = \exp\left(\int p(t)\,dt\right) = e^{\int p(t)\,dt}$

Lesson 3

So we get that

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y].$$

Since $\frac{dy}{dt} + p(t)y = g(t)$,

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t).$$

So we get

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

Integrating both sides (using Fund. Thm. of Calc)

$$\mu(t)y = \int \mu(t)g(t)\,dt + C$$

We can then solve for y to get

$$y(t) = \frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)g(s)\,ds + \frac{c}{\mu(t)}$$

for some convenient choice of t_0.

Sometimes $\int_{t_0}^{t}\mu(s)g(s)\,ds$ is not solvable using familiar functions, so we can leave the solution in this form in that case.

We will often deal with functions when the integral <u>can</u> be solved, however.

It is also easier to simply learn the formula for $\mu(t)$. Below is a general outline for how to use this method effecientlly.

Lesson 3

Method of Integrating Factors:

1. Given a first order linear ODE, write it in the form $y' + p(t)y = g(t)$
2. Compute the integrating factor $\mu(t) = \exp(\int p(t)\,dt)$
3. Have the formula $\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$.
4. Integrate both sides (don't forget $+C$)
$$\mu(t)y = \int \mu(t)g(t)\,dt + C$$
5. Solve for y.

Ex 1. Solve the diff eq and describe the behavior of solutions as $t \to \infty$.

$$ty' + y = 3t\cos 2t,\ t > 0$$
$$y' + \frac{1}{t}y = 3\cos 2t$$
$$\mu(t) = \exp\left(\int \frac{1}{t}\,dt\right) = e^{\ln|t|} = |t| = t \text{ since } t > 0$$
$$\frac{d}{dt}[ty] = 3t\cos 2t$$

Use integration by parts: $u = 3t$ $\quad v = \frac{1}{2}\sin 2t$

$du = 3\,dt$ $\quad dv = \cos 2t\,dt$

$$ty = \frac{3}{2}t\sin 2t - \int \frac{3}{2}\sin 2t\,dt$$
$$ty = \frac{3}{2}t\sin 2t + \frac{3}{4}\cos 2t + C$$
$$\boxed{y(t) = \frac{3}{2}\sin 2t + \frac{3\cos 2t}{4t} + \frac{C}{t}}$$

as $t \to \infty$: $\frac{3\cos 2t}{4t} \to 0$, $\frac{C}{t} \to 0$

So as $t \to \infty$, y is asymptotic to $\frac{3}{2}\sin 2t$.

Lesson 3

Ex 2. Consider $2y' - y = e^{t/3}$, $y(0) = a$.

(a) Draw a direction field and see if behavior depends on a.

→ use dfield8

$$y' = \tfrac{1}{2}y + \tfrac{1}{2}e^{t/3}$$

does depend!

(b) Solve the equation and find the value a_0.

$$y' - \tfrac{1}{2}y = \tfrac{1}{2}e^{t/3}$$

$$\mu(t) = \exp\left(\int -\tfrac{1}{2}\,dt\right) = e^{-t/2}$$

$$\frac{d}{dt}\left[e^{-t/2}y\right] = \tfrac{1}{2}e^{-t/6}$$

$$e^{-t/2}y = -3e^{-t/6} + C$$

$$y(t) = -3e^{t/3} + Ce^{t/2}$$

$$a = y(0) = -3 + C, \text{ so } C = a + 3$$

$$y(t) = -3e^{t/3} + (a+3)e^{t/2}$$

$a_0 = -3$, since if $a_0 > -3$, $(a+3)e^{t/2}$ is positive and dominates; if $a_0 < -3$, $(a+3)e^{t/2}$ is negative

Lesson 3

Ex 3. Consider $y' + \frac{1}{2}y = 2 - t$, $y(0) = y_0$. Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

$$\mu(t) = \exp\left(\int \tfrac{1}{2}\,dt\right) = e^{t/2}$$
$$\frac{d}{dt}\left[e^{t/2}y\right] = 2e^{t/2} - te^{t/2}$$
$$u = t \qquad v = 2e^{t/2}$$
$$du = dt \qquad dv = e^{t/2}dt$$
$$e^{t/2}y = 4e^{t/2} - 2te^{t/2} + \int 2e^{t/2}dt$$
$$e^{t/2}y = 4e^{t/2} - 2te^{t/2} + 4e^{t/2} + C$$
$$y(t) = 8 - 2t + Ce^{-t/2}$$
$$y_0 = y(0) = 8 + C, \text{ so } C = y_0 - 8$$
$$y(t) = 8 - 2t + (y_0 - 8)e^{-t/2}$$

In order to touch the t-axis but not cross, must have a min or max touching t-axis. Find t-value where min or max occurs.

$$0 \overset{\text{set}}{=} y'(t) = -2 - \tfrac{1}{2}(y_0 - 8)e^{-t/2}$$
$$\frac{-4}{y_0 - 8} = e^{-t/2}$$
$$\text{so } t = -2\ln\left(\frac{-4}{y_0 - 8}\right)$$

Need $y(t) = 0$ at this t-value

$$0 \overset{\text{set}}{=} 8 - 2\left(-2\ln\left(\frac{-4}{y_0-8}\right)\right) + (y_0 - 8)e^{-\left(-2\ln\left(\frac{-4}{y_0-8}\right)\right)/2}$$

Use Wolfram Alpha to find

$$y_0 = 8 - 4e \approx -2.8731$$

For a sanity check, graph

$$y(t) = 8 - 2t + (8 - 4e - 8)e^{-t/2}$$

Lesson 3

Ex 4. Show that all solutions of $4y' + ty = 4$ approach a limit as $t \to \infty$ and find the limiting value.

$$y' + \frac{t}{4}y = 1$$

$$\mu(t) = \exp\left(\int \frac{t}{4}\,dt\right) = e^{t^2/8}$$

$$\frac{d}{dt}\left[e^{t^2/8}y\right] = e^{t^2/8}$$

$$e^{t^2/8}y = \int e^{t^2/8}\,dt + c \leftarrow \text{can't integrate with normal functions!}$$

$$y = \underbrace{e^{-t^2/8}}_{\to 0}\underbrace{\int_{t_0}^{t} e^{s^2/8}\,ds}_{\to \infty} + \underbrace{ce^{-t^2/8}}_{\to 0}$$

as $t \to \infty$,

In determinate form $0 \cdot \infty$

Take $\displaystyle\lim_{t\to\infty} \frac{\int_{t_0}^{t} e^{s^2/8}\,ds}{e^{t^2/8}}$ $\quad\left(\frac{\infty}{\infty} \text{ form}\right)$

Apply L'Hopital's Rule.

By the Fundamental Theorem of Calculus,

$$\frac{d}{dt}\left(\int_{t_0}^{t} e^{s^2/8}\,ds\right) = e^{t^2/8}$$

get $\displaystyle\lim_{t\to\infty} \frac{e^{t^2/8}}{\frac{t}{4}e^{t^2/8}} = \lim_{t\to\infty} \frac{1}{\frac{t}{4}}$

$$= \lim_{t\to\infty} \frac{4}{t} = 0$$

So $\displaystyle\lim_{t\to\infty} y(t) = 0$ for all solutions $y(t)$.