

## Lesson 33

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### Complex Eigenvalues (7.6)

If  $A$  is a real  $2 \times 2$  matrix and  $\vec{u} + i\vec{v}$  is a solution to  $\vec{x}' = A\vec{x}$  ( $\vec{u}$  and  $\vec{v}$  are real 2-vectors) then both  $\vec{u}$  and  $\vec{v}$  are solutions as well.  
( $\vec{u}' + i\vec{v}' = A\vec{u} + iA\vec{v}$ , so  $\vec{u}' = A\vec{u}$  and  $\vec{v}' = A\vec{v}$ )

That being said, we can express our solution in terms of real values given one eigenvalue  $\lambda$  and one eigenvector  $\vec{\xi}$  (when we have complex conjugate eigenvalues). To be clear this is only true when  $A$  is a real matrix (all entries are real numbers).

To find the solution of  $\vec{x}' = A\vec{x}$  when  $A$  is real and has complex eigenvalues  $\lambda \pm i\mu$ , find an eigenvector  $\vec{\xi}$  associated with  $\lambda + i\mu$ .

Then one solution is of the form

$$\vec{\xi} e^{(\lambda + i\mu)t} = \vec{\xi} (e^{\lambda t} [\cos(\mu t) + i \sin(\mu t)])$$

Distribute into the vector  $\vec{\xi}$ .

Separate into two vectors  $\vec{u} + i\vec{v}$ .

Then the solutions are of the form

$$c_1 e^{\lambda t} \vec{u} + c_2 e^{\lambda t} \vec{v}.$$

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Ex 1. Complex eigenvalues with positive real part.

$$\vec{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \vec{x}$$

Find eigenvalues:

$$\begin{vmatrix} 2-\lambda & -\frac{5}{2} \\ \frac{9}{5} & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - \left(\frac{9}{5}\right)\left(-\frac{5}{2}\right) = \lambda^2 - \lambda - 2 + \frac{9}{2}$$

$$\lambda = \frac{1 \pm \sqrt{1-10}}{2} = \frac{1 \pm 3i}{2} = \frac{1}{2} \pm \frac{3}{2}i$$

$$\begin{pmatrix} \frac{3}{2} - \frac{3}{2}i & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} - \frac{3}{2}i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \left(\frac{3}{2} - \frac{3}{2}i\right)x_1 = \frac{5}{2}x_2 \\ \left(\frac{9}{5} - \frac{3}{5}i\right)x_1 = x_2 \end{matrix} \quad \vec{v} = \begin{pmatrix} 5 \\ 3-3i \end{pmatrix}$$

$$\begin{aligned} & \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{t/2} \left( \cos\left(\frac{3}{2}t\right) + i \sin\left(\frac{3}{2}t\right) \right) \\ &= e^{t/2} \begin{pmatrix} 5 \cos\left(\frac{3}{2}t\right) + 5i \sin\left(\frac{3}{2}t\right) \\ 3 \cos\left(\frac{3}{2}t\right) + 3i \sin\left(\frac{3}{2}t\right) - 3i \cos\left(\frac{3}{2}t\right) + 3 \sin\left(\frac{3}{2}t\right) \end{pmatrix} \\ &= e^{t/2} \begin{pmatrix} 5 \cos\left(\frac{3}{2}t\right) \\ 3 \cos\left(\frac{3}{2}t\right) + 3 \sin\left(\frac{3}{2}t\right) \end{pmatrix} + e^{t/2} i \begin{pmatrix} 5 \sin\left(\frac{3}{2}t\right) \\ 3 \sin\left(\frac{3}{2}t\right) - 3 \cos\left(\frac{3}{2}t\right) \end{pmatrix} \end{aligned}$$

so

$$\vec{x}(t) = c_1 e^{t/2} \begin{pmatrix} 5 \cos\left(\frac{3}{2}t\right) \\ 3 \cos\left(\frac{3}{2}t\right) + 3 \sin\left(\frac{3}{2}t\right) \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 5 \sin\left(\frac{3}{2}t\right) \\ 3 \sin\left(\frac{3}{2}t\right) - 3 \cos\left(\frac{3}{2}t\right) \end{pmatrix}$$

Use plane 8 to graph.

Notice trajectories are all spirals, so  $\vec{0}$  is called a spiral point.

As  $t \rightarrow \infty$ ,  $\vec{x}(t)$  diverges from  $\vec{0}$ , so  $\vec{0}$  is an unstable spiral point.

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Ex 2. Complex eigenvalues with negative real part

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) - (5)(-1) = \lambda^2 + 2\lambda - 3 + 5$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2-i)x_1 = x_2 \quad \vec{v} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$e^{-t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} [\cos(t) + i \sin(t)]$$

$$= e^{-t} \begin{pmatrix} \cos(t) + i \sin(t) \\ 2\cos(t) + 2i \sin(t) - i \cos(t) + \sin(t) \end{pmatrix}$$

$$= e^{-t} \left[ \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right]$$

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Plot with plane 8.

As  $t \rightarrow \infty$ , trajectories converge to  $\vec{0}$ .

Trajectories are spirals toward  $\vec{0}$ ,

So  $\vec{0}$  is an asymptotically stable spiral point.

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Ex 3. Purely imaginary eigenvalues

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (1)(-5) = \lambda^2 - 4 + 5 = \lambda^2 + 1$$

$$\lambda = \pm i$$

$$\left( \begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array} \right); \quad \begin{array}{l} (2-i)x_1 = 5x_2 \\ \frac{2-i}{5}x_1 = x_2 \end{array} \quad \vec{v} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} [\cos(t) + i\sin(t)] &= \begin{pmatrix} 5\cos t + 5i\sin t \\ 2\cos t + 2i\sin t - i\cos t + \sin t \end{pmatrix} \\ &= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} \end{aligned}$$

$$\boxed{\vec{x}(t) = C_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}}$$

Plot with pplane8

All trajectories are elliptical, so  $\vec{0}$  is called a center.

As  $t \rightarrow \infty$ , the trajectories neither converge nor diverge, so we say  $\vec{0}$  is stable (but not asymptotically stable).

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## Critical values for phase portraits:

We obtain critical values when the behavior changes from one form to another.

- Changes from complex to real or vice versa  
(nodes or saddle points to spiral or center points)
- Changes from complex to purely imaginary or vice versa  
(spiral points to center points)
- Eigenvalue = 0.  
(changes from one type to another)

Ex 4. Find the critical values of  $\alpha$  where the qualitative nature of the phase portrait changes.

$$\vec{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -\lambda & -5 \\ 1 & \alpha - \lambda \end{vmatrix} = (-\lambda)(\alpha - \lambda) - (1)(-5) = \lambda^2 - \alpha\lambda + 5$$

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 20}}{2}$$

Possibilities: Changes from complex to real or vice versa

occurs when  $\alpha^2 - 20 = 0$  i.e., when  $\alpha = \pm\sqrt{20}$

Eigenvalues are complex when  $-\sqrt{20} < \alpha < \sqrt{20}$

When  $\lambda = 0$ :

$$\alpha - \sqrt{\alpha^2 - 20} = 0$$

$$\alpha = \sqrt{\alpha^2 - 20}$$

$$\alpha^2 = \alpha^2 - 20$$

$$0 = -20$$

→ This type never occurs

Purely imaginary eigen values occur when  $\alpha = 0$ .

So critical values are

$$\alpha = -\sqrt{20} \text{ or } 0 \text{ or } \sqrt{20}$$