# Lesson 35

## Nonhomogeneous Systems (7.9)

A system of equations $\vec{x}' = A\vec{x} + \vec{g}$ is nonhomogeneous if $\vec{g} \neq \vec{0}$.

There are many ways to solve such systems. There is a method involving diagonalization of matrices. You can use the Laplace Transform (picking initial conditions and extracting the general solution). You can use Variation of Parameters (discussed in MA 303 and MA 304). We start with Undetermined Coefficients.

## Method of Undetermined Coefficients.

Mainly the same as with 2nd order equations.

1. Find complementary solution $\vec{x}_c(t)$.
2. Looking at $\vec{g}(t)$, guess the form for a particular solution (using arbitrary vectors)
3. If part of your solution is in $\vec{x}_c(t)$, adjust*.
4. $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$

* if some part of your guess is in $\vec{x}_c(t)$, multiply by $t$, but like with repeated eigenvalues, we need an extra term without $t$ (like we needed $\vec{\eta}$)

Ex: if original guess is $\vec{a}e^{\alpha t}$, and this in $\vec{x}_c(t)$, get $\vec{a}te^{\alpha t} + \vec{b}e^{\alpha t}$

Ex 1. Find an appropriate form for a particular solution $\vec{x}_p(t)$ to the following system

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Find complementary solution:

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+5 = \lambda^2+1; \quad \lambda = \pm i$$

$$\left(\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right) \quad (2-i)x_1 - 5x_2 = 0 \quad \text{so } x_2 = \frac{2-i}{5}x_1 \qquad \text{choose } x_1 = 5,\ x_2 = 2-i$$

$$\vec{\xi} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos t + i\,5\sin t \\ 2\cos t - i\cos t + i\,2\sin t + \sin t \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

$$\vec{x}_c(t) = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t$$

terms: $\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t$

guess 1: $\vec{a}\cos t + \vec{b}\sin t$

in $x_c$?: yes (notice $x_c$ could be rewritten as $\vec{\alpha}\cos t + \vec{\beta}\sin t$)

guess 2: $\underbrace{\vec{a}t\cos t + \vec{b}t\sin t}_{\text{multiply by } t} + \underbrace{\vec{c}\cos t + \vec{d}\sin t}_{\text{extra adjusting terms}}$

$$\boxed{\vec{x}_p(t) = \vec{a}t\cos t + \vec{b}t\sin t + \vec{c}\cos t + \vec{d}\sin t}$$

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Ex 2. Find an appropriate form for $\vec{x}_p(t)$ for

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

Find complementary solution:

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+3 = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$\lambda_1 = 1$: $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right)$ $\quad x_1 - x_2 = 0$ i.e. $x_1 = x_2$ $\quad$ choose $x_1 = 1$, $x_2 = 1$ $\quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$: $\left(\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right)$ $\quad 3x_1 - x_2 = 0$ i.e., $x_2 = 3x_1$ $\quad$ choose $x_1 = 1$, $x_2 = 3$ $\quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\vec{x}_c(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

| | | |
|---|---|---|
| terms: | $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix} t$ |
| guess 1: | $\vec{a}e^t$ | $\vec{c}t + \vec{d}$ |
| in $x_c$?: | yes | no |
| guess 2: | $\underbrace{\vec{a}te^t}_{\text{times } t} + \underbrace{\vec{b}e^t}_{\text{adjusting term}}$ | same as guess 1 |

$$\boxed{\vec{x}_p(t) = \vec{a}te^t + \vec{b}e^t + \vec{c}t + \vec{d}}$$

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Ex 3. Find the general solution

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$\lambda_1 = 1:$ $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right)$ $\quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1:$ $\left(\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right)$ $\quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\vec{x_c}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

terms: $\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

guess 1: $\vec{a} e^t$

in $\vec{x_c}$?: yes

guess 2: $\underbrace{\vec{a} t e^t}_{\text{times } t} + \underbrace{\vec{b} e^t}_{\text{adjusting term}}$

$$\vec{x_p}(t) = \vec{a} t e^t + \vec{b} e^t$$

$$\vec{x_p}'(t) = \vec{a} t e^t + \vec{a} e^t + \vec{b} e^t$$

$$A\vec{x_p}(t) + \vec{g}(t) = A\vec{a}\, t e^t + A\vec{b} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$A\vec{a} = \vec{a}$ $\rightarrow$ $\vec{a}$ is an eigenvector of $A$ assoc. to 1

i.e., $\vec{a} = c\,\vec{\xi}^{(1)} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix}$ for some constant $c$.

$$\vec{a} + \vec{b} = A\vec{b} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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$$\begin{pmatrix} c \\ c \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} c+b_1 \\ c+b_2 \end{pmatrix} = \begin{pmatrix} 2b_1 - b_2 + 1 \\ 3b_1 - 2b_2 - 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 - b_2 - c \\ 3b_1 - 3b_2 - c \end{pmatrix}$$

$$\begin{aligned} b_1 - b_2 - c &= -1 \\ 3b_1 - 3b_2 - c &= 1 \end{aligned} \Rightarrow \begin{aligned} -3b_1 + 3b_2 + 3c &= 3 \\ 3b_1 - 3b_2 - c &= 1 \\ \hline 2c &= 4 \\ c &= 2 \end{aligned}$$

Then $\begin{aligned} b_1 - b_2 &= 1 \\ 3b_1 - 3b_2 &= 3 \end{aligned}$ (same line).

Choose $b_2 = 0$, so $b_1 = 1$

$$\vec{a} = \begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}_p(t) = \vec{a}te^t + \vec{b}e^t = te^t\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t$$

$$\boxed{\vec{x}(t) = \underbrace{c_1 e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\vec{x_c}(t)} + \underbrace{te^t\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t}_{\vec{x_p}(t)}}$$

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## Variation of Parameters

Given a system $\vec{x}' = A\vec{x} + \vec{g}$,

find a fundamental matrix $\Phi(t)$ for $\vec{x}' = A\vec{x}$.

Assume the general solution is $\vec{x}(t) = \Phi(t)\vec{u}(t)$ for some vector $\vec{u}$.

Then $\Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = \underbrace{A\,\Phi(t)}_{\Phi'(t)}\vec{u}(t) + \vec{g}(t)$

$$\Phi(t)\vec{u}'(t) = \vec{g}(t)$$

Get the system of equations

$$\Phi(t)\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = g(t)$$

Solve for $u_1, u_2$

Ex 4. $\vec{x}'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t$

Already seen $\lambda_1 = 1$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1$, $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

so $\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$

$\Phi(t)\vec{u}'(t) = g(t)$

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$e^t u_1' + e^{-t} u_2' = e^t$$
$$e^t u_1' + 3e^{-t} u_2' = -e^t$$

Multiply first equation by $e^t$:

$e^{2t}u_1' + u_2' = e^{2t}$,

so $u_2' = e^{2t}(1-u_1')$

Plug into 2nd equation:

$$e^t u_1' + 3e^{-t}e^{2t}(1-u_1') = -e^{-t}$$
$$e^t u_1' + 3e^t(1-u_1') = -e^{-t}$$
$$e^t u_1' - 3e^t u_1' = -e^{-t} - 3e^t$$
$$-2e^t u_1' = -e^{-t} - 3e^t$$
$$u_1' = -\tfrac{1}{2}(-e^{-2t} - 3) = \frac{e^{-2t}}{2} + \frac{3}{2}$$
$$u_1 = \frac{e^{-2t}}{-4} + \frac{3}{2}t + c_1$$

Multiply 2nd equation by $e^{-t}$:

$$u_1' + 3e^{-2t}u_2' = -e^{-2t}$$
$$u_1' = -e^{-2t} - 3e^{-2t}u_2'$$

Plug into 1st equation:

$$e^t(-e^{-2t} - 3e^{-2t}u_2') + e^{-t}u_2' = e^t$$
$$-e^{-t} - 3e^{-t}u_2' + e^{-t}u_2' = e^t$$
$$-2e^{-t}u_2' = e^t + e^{-t}$$
$$u_2' = -\tfrac{1}{2}(e^{2t}+1) = -\tfrac{1}{2}e^{2t} - \tfrac{1}{2}$$
$$u_2 = -\tfrac{1}{4}e^{2t} - \tfrac{1}{2}t + c_2$$

$$\vec{x}(t) = \Phi(t)\vec{u}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} -\frac{1}{4}e^{-2t} + \frac{3}{2}t + c_1 \\ -\frac{1}{4}e^{2t} - \frac{1}{2}t + c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4}e^{-t} + \frac{3}{2}te^t + \underline{c_1e^t} - \frac{1}{4}e^t - \frac{1}{2}te^{-t} + \underline{c_2e^{-t}} \\ -\frac{1}{4}e^{-t} + \frac{3}{2}te^t + \underline{c_1e^t} - \frac{3}{4}e^t - \frac{3}{2}te^{-t} + \underline{3c_2e^{-t}} \end{pmatrix}$$

$$= \underline{c_1e^t\begin{pmatrix}1\\1\end{pmatrix}} + \underline{c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}} + \begin{pmatrix}\frac{3}{2}\\\frac{3}{2}\end{pmatrix}te^t + \begin{pmatrix}\frac{1}{4}\\-\frac{1}{4}\end{pmatrix}e^t + \begin{pmatrix}-\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}te^{-t}$$