1. Consider a tank which has 400 gallons of a salt-water mixture. Initially, the tank has 15 lbs of salt in it. Water flows into the tank at a rate of 30 gallons per minute, and there is 1/2 lb of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the salt-water mixture. The salt-water mixture flows out of the tank at a rate of 30 gallons per minute. Find the concentration (lbs per gallon) of salt in the tank in the long run.

Let $q(t)$ be the amount of salt in the tank after t minutes.
Let $c(t)$ be the concentration of salt after t minutes.
Let $V(t)$ be the volume of liquid in the tank after t minutes.

$$\frac{dq}{dt}\left(\frac{\text{lbs}}{\text{min}}\right) = \text{rate in}\left(\frac{\text{lbs}}{\text{min}}\right) - \text{rate out}\left(\frac{\text{lbs}}{\text{min}}\right)$$

rate in: $\frac{\frac{1}{2}\text{ lb}}{\text{gal}} \cdot \frac{30\text{ gal}}{\text{min}} = 15\ \frac{\text{lbs}}{\text{min}}$; rate out: $\frac{q(t)\text{ lbs}}{V(t)\text{ gal}} \cdot \frac{30\text{ gal}}{\text{min}}$

$V(t) = 400$ for all t $\left(30\ \frac{\text{gal}}{\text{min}}\text{ in}, 30\ \frac{\text{gal}}{\text{min}}\text{ out}\right)$

$$\frac{dq}{dt} = 15 - \frac{30}{400}q \quad \text{(integrating factor or separable)}$$

get $q(t) = Ce^{-3t/40} + 200$

$q(0) = 15$, so $15 = C + 200 \Rightarrow C = -185$

$$q(t) = -185e^{-3t/40} + 200$$

$$c(t) = \frac{q(t)}{V(t)} = \frac{-185e^{-3t/40} + 200}{400}$$

$$= -\frac{37}{80}e^{-3t/40} + \frac{1}{2}$$

$$\lim_{t\to\infty} c(t) = 0 + \frac{1}{2} = \frac{1}{2}$$

$$\boxed{\frac{1}{2}\text{ lb per gal}}$$

2. Consider a tank which has 400 gallons of pure water, and has a capacity of 700 gallons. Salt water begins to flow into the tank at a rate of 5 gallons per minute and there are 10 grams of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the tank. The mixture in the tank flows out at a rate of 3 gallons per minute. How much salt will be in the tank the instant it begins to overflow?

Use same symbols as problem 1.

Notice: $V(t)$ is not constant.

$$\frac{dV}{dt} = \text{rate in} - \text{rate out} = 5\,\frac{\text{gal}}{\text{min}} - 3\,\frac{\text{gal}}{\text{min}} = 2\,\frac{\text{gal}}{\text{min}}$$

$$V(t) = 2t + C,\quad V(0) = 400,\ \text{so}\ V(t) = 2t + 400$$

$$\frac{dq}{dt} = \text{rate in} - \text{rate out} = \frac{10\,g}{\text{gal}} \cdot \frac{5\,\text{gal}}{\text{min}} - \frac{q(t)\,g}{\text{gal}} \cdot \frac{3\,\text{gal}}{\text{min}}$$

$$\frac{dq}{dt} = 50 - \frac{3q}{2t+400}$$

Use integrating factors: $q(t) = \dfrac{C}{(t+200)^{3/2}} + 20t + 4000$

$q(0) = 0$ (pure water) get $C = -8{,}000{,}000\sqrt{2}$

$$q(t) = \frac{-8{,}000{,}000\sqrt{2}}{(t+200)^{3/2}} + 20t + 4000$$

Tank overflows when $V(t) = 700$

$$700 = 2t + 400 \Rightarrow t = 150$$

$$q(150) \approx \boxed{5{,}272.16 \text{ grams}}$$