1. A construction worker stands at the top of a 40 meter bell tower and throws a clock face upward with a speed of 5 m/s. Suppose there is a force due to air resistance acting on the clock face in the opposite direction of velocity with a magnitude of $\frac{|v|}{22}$ m/s. Set up a differential equation to model this scenario (use g = 9.8 m/s^2 as the magnitude of the acceleration due to gravity).

Let $v(t)$ be velocity at t seconds

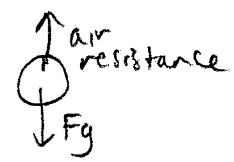
Let m be the mass of the clock face.

Let $\uparrow$ be positive direction.

$$F = ma = m\frac{dv}{dt}$$

going up:

going down:



In either case, $-v$ is opposite to velocity

air resistance: $-\frac{v}{22}$, $F_g = -mg$

$$-mg - \frac{v}{22} = m\frac{dv}{dt}$$

$$\frac{dv}{dt} = -g - \frac{v}{22m} = -9.8 - \frac{v}{22m},$$

initial conditions: $v(0) = 5$, $h(0) = 40$

2. Pete stands on the top of a 20 foot train and throws a 2 slug hammer upward with a speed of 10 ft/s. Suppose there is a force due to air resistance acting on the hammer in the opposite direction of velocity with a magnitude of $\frac{v^2}{2000}$ ft/s. Assuming the hammer misses the train, how long will it take the hammer to hit the ground? (Use g = 32 ft/s^2 as the magnitude of the acceleration due to gravity.)

Same basic set up as problem 1

Need two separate equations for air resistance to be opposite velocity!
($v>0$ needs air resist <0 and vice versa)

up: $m\frac{dv}{dt} = -mg - \frac{v^2}{2000}$, down: $m\frac{dv}{dt} = -mg + \frac{v^2}{2000}$

First, going up: $\frac{dv}{dt} = -32 - \frac{v^2}{4000} = \frac{-v^2 - 128{,}000}{4000}$

$$\frac{dv}{-128{,}000 - v^2} = \frac{dt}{4000}$$

$\left(\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C \right)$

$$-\frac{1}{128{,}000}\,\frac{dv}{1+\left(\frac{v}{\sqrt{128{,}000}}\right)^2} = \frac{dt}{4000}$$

$$-\frac{\sqrt{128{,}000}}{128{,}000}\tan^{-1}\left(\frac{v}{\sqrt{128{,}000}}\right) = \frac{t}{4000} + C$$

Using initial condition $v(0) = 10$ and using algebra

$$v(t) = \sqrt{128{,}000}\,\tan\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)$$

Maximum height occurs when $v(T) = 0$
Solving for T, get $T \approx 0.312418658$ seconds

Now, $\frac{dh}{dt} = v$, so $h(t) = \int v(t)\,dt$
(Recall $\int \tan(u)\,du = \ln|\sec u| + C$)

$$h(t) = -4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)\right| + C$$

Using $h(0) = 20$

$$h(t) = -4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)\right| + \left(4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right)\right)\right| + 20\right)$$

max height occurs at T found above
$h(T) \approx 21.561889966$ feet.

Going down: Reset to $t=0$ for simplicity.

$$m\frac{dv}{dt} = -mg + \frac{v^2}{2000}, \quad v(0)=0, \quad h(0) \approx 21.561889966$$

$$\frac{dV}{-128{,}000+V^2} = \frac{dt}{4000} \qquad \left(\int \frac{1}{1-u^2}\,du = \tanh^{-1}(u) + C \text{ for } -1<u<1 \right)$$

$$-\frac{1}{128{,}000} \cdot \frac{dV}{1-\left(\frac{V}{\sqrt{128{,}000}}\right)^2} = \frac{dt}{4000}$$

$$-\frac{\sqrt{128{,}000}}{128{,}000} \tanh^{-1}\left(\frac{V}{\sqrt{128{,}000}}\right) = \frac{t}{4000} + C$$

Using $v(0)=0$, algebra, we get

$$v(t) = \sqrt{128{,}000}\tanh\left(-\frac{\sqrt{128{,}000}}{4{,}000}t\right)$$

$$h(t) = \int v(t)\,dt \qquad \left(\int \tanh(u)\,du = \ln|\cosh(u)| + C \right)$$

$$h(t) = -4000\ln\left|\cosh\left(-\frac{\sqrt{128{,}000}}{4000}t\right)\right| + C$$

Using $h(0) \approx 21.561889966$

$$h(t) = -4000\ln\left|\cosh\left(-\frac{\sqrt{128{,}000}}{4000}t\right)\right| + 21.561889966$$

Ball hits the ground when $h(S)=0$

Solving to get $S \approx 1.1619127$

Time to go up + time to go down

$$T + S$$

$$\approx 0.31241865 8 + 1.1619127$$

$$\approx \boxed{1.47 \text{ seconds}}$$

3. Suppose that a rocket is launched straight up from the surface of the Earth with an initial velocity of $v_0 = \sqrt{2gR}$, where R is the radius of the Earth. Neglect air resistance. Find an expression for the velocity v in terms of the distance x from the surface of the Earth. Find the time required for the rocket to go 140,000,000 miles (the approximate distance from Earth to Mars). Assume that $R = 4000$ miles. (There are 5280 feet in a mile.)

Such a situation is governed by the diff eq

$$\frac{dv}{dt} = \frac{-gR^2}{(R+x)^2}$$

By the chain rule, $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$ $\left(\frac{dx}{dt} = v\right)$

$$v\frac{dv}{dx} = \frac{gR^2}{(R+x)^2}, \quad v(0) = \sqrt{2gR}$$

get $v(x) = \frac{R\sqrt{2g}}{\sqrt{R+x}}$

$\frac{dx}{dt} = v(t)$, so $\frac{dx}{dt} = \frac{R\sqrt{2g}}{\sqrt{R+x}}$

$$(R+x)^{1/2}\,dx = R\sqrt{2g}\,dt$$

$$\frac{2}{3}(R+x)^{3/2} = Rt\sqrt{2g} + C$$

$x(0) = 0$

$$\frac{2}{3}(R+x)^{3/2} = Rt\sqrt{2g} + \frac{2}{3}R^{3/2}$$

To solve for time, use $x = 140{,}000{,}000$, $R = 4000$, $g = \frac{32\text{ ft}}{s^2} \cdot \frac{1\text{ mi}}{5280\text{ ft}}$

This gives time in seconds

convert to hours

$\approx 696{,}600.57$ hrs

(≈ 79.5 years)

4. Suppose that the rate of change of a function f is proportional to a function g. Write a differential equation which expresses this situation.

$$f' = kg$$

for some constant k

5. If Jack weighs 200 lbs, what is his mass?

$$200 = mg \qquad (g = 32\ ft/s^2)$$

$$m = \frac{200}{32} = 6.25 \text{ slugs}$$