MA 265, Lesson 18

Rank (4.9)

Recall:

Thm 4.18 The row rank and column rank of a matrix are equal.

Thm 4.19 If A is an $m \times n$ matrix, then rank A + nullity $A = n$.

(You will be asked to verify these theorems)

Now, for an $n \times n$ matrix, rank is related to singularity (and other related things.)

Theorem 4.20 If A is an $n \times n$ matrix, then rank $A = n$ if and only if A is row equivalent to I_n.

Proof: Transform A into RREF. If rank $A = n$, then A has n nonzero rows. Since A has n rows in total, $RREF(A) = I_n$. Thus, A is row equivalent to I_n.
If A is row equivalent to I_n, then $RREF(A) = I_n$. Since rank A is the number of nonzero rows of $RREF(A) = I_n$, rank $A = n$. ▨

Being row equivalent to I_n gives us a lot of information.

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<u>Corollary 4.7</u> An $n \times n$ matrix A is nonsingular if and only if $\operatorname{rank} A = n$.

<u>Corollary 4.8</u> If A is an $n \times n$ matrix, then $\operatorname{rank} A = n$ if and only if $\det(A) \neq 0$.

<u>Corollary 4.9</u> Let A be $n \times n$. The homogeneous system $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if $\operatorname{rank} A < n$.

<u>Corollary 4.10</u> Let A be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a unique solution for every $n \times 1$ matrix $\vec{b}$ if and only if $\operatorname{rank} A = n$.

These are proven using Thm 4.20 and already established equivalences.

<u>Ex. 1.</u> Show that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ spans R^3.

Form the matrix:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\operatorname{rank} A = 3$, so $\det(A) \neq 0$, so columns are linearly independent, so since $\dim R^3 = 3$, also spanning.

<u>Theorem 4.21</u> The linear system $A\vec{x} = \vec{b}$ has a solution if and only if $\operatorname{rank} A = \operatorname{rank} [A \mid \vec{b}]$.

<u>Proof</u>. Suppose $\vec{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is a solution to the system $A\vec{x} = \vec{b}$. Let $\vec{x}_1, \dots, \vec{x}_n$ be the columns of A. Then $\vec{b} = a_1\vec{x}_1 + \dots + a_n\vec{x}_n$, so $\vec{b}$ is a linear combination of the columns of A. As such, converting $[A \mid \vec{b}]$ into RCEF, we get $[C \mid \vec{0}]$ where $C = RCEF(A)$. Thus, $\operatorname{rank} A = \operatorname{rank} C =$ column rank $[C \mid \vec{0}] = \operatorname{rank} [C \mid \vec{0}]$
$= \operatorname{rank} [A \mid \vec{b}]$.

Reversing the argument gives the other direction. $\square$

Ex 2. Use Thm 4.21 to show

$$\begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$ is inconsistent.

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 4 & -1 & -5 & 6 & 2 \\ 2 & 3 & 1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 8/7 & 0 \\ 0 & 1 & 1 & -10/7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ rank } 3$$

$\neq$

$$\begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -1 & -5 & 6 \\ 2 & 3 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 8/7 \\ 0 & 1 & 1 & -10/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ rank } 2$$

Recall $\operatorname{rank} A \leq \min\{m, n\}$ when A is $m \times n$.
(since row rank $A \leq m$ and column rank $A \leq n$).

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Some things you can show with rank!

<u>Ex 3.</u> Let A be $m \times n$. Show that $A\vec{x} = \vec{b}$ has a solution for every $m \times 1$ matrix $\vec{b}$ if and only if rank $A = m$.

<u>Proof.</u> If $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$, by Thm 4.20, $[A \mid \vec{b}]$ has the same rank as A for every $\vec{b}$. Thus, every vector in $\mathbb{R}^n$ is a linear combo of the columns of A, so the columns of A span $\mathbb{R}^n$. This is not possible if rank $A \neq m$. Conversely, if rank $A = m$, then every row is nonzero for A in RREF, so we get a solution.

<u>Ex 3.</u> Let A be $m \times n$ with $m \neq n$. Show that either the rows or columns of A are linearly dependent.

<u>Proof.</u> rank $A \leq \min\{m, n\}$.

<u>case 1</u>. Suppose $m < n$. Then rank $A \leq m < n$. Putting A in RCEF, we get $\leq m$ nonzero rows, thus dim column space $A \leq m < n$, so the columns of A are linearly dependent (more than in a basis).

<u>case 2</u>. Suppose $n < m$. Then rank $A \leq n < m$. Putting A in RREF, we get $\leq n$ nonzero rows, thus dim row space $A \leq n < m$, so the rows of A are linearly dependent. ▨

Ex 4. Show rank A = rank A^T. Does nullity A = nullity A^T?

Proof. rank A = row rank A = column rank A^T = rank A^T. $\boxtimes$

Now, nullity $A \neq$ nullity A^T in general.
If A is 4×7 and rank $A = 4$, then nullity $A = 3$.
Now, A^T is 7×4 and rank $A^T = 4$, so nullity $A^T = 0$.

Ex 5. Suppose $A\vec{x} = \vec{b}$ is consistent (A is $m \times n$).
Prove the solution is unique if and only if rank $A = n$.

Proof. Suppose the solution is unique. Let $\vec{x_p}$ be the particular solution. Since $\vec{x_p} + \vec{x_h}$ is a solution for any homogeneous solution $\vec{x_h}$, $\vec{x_h} = \vec{0}$ is the only homogeneous solution. Thus nullity $A = 0$.
By Rank-Nullity, rank $A = n - 0 = n$.
Suppose rank $A = n$. Then by rank-nullity, nullity $A = 0$. So $\vec{0}$ is the only homogeneous solution. Since all solutions to $A\vec{x} = \vec{b}$ are of the form $\vec{x_p} + \vec{x_h}$, $\vec{x_p} + \vec{0} = \vec{x_p}$ is the only solution, so it is unique. $\boxtimes$