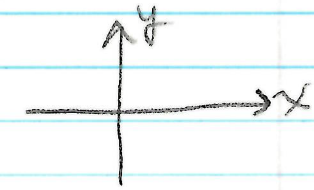
MA 261 – Lesson 1

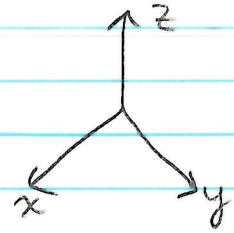
Geometry of Space + vectors

12.1:

$\mathbb{R}^2$ represents 2-dimensional space



$\mathbb{R}^3$ represents 3-dimensional space

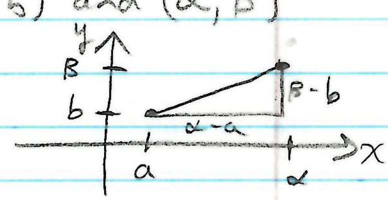


3-dimensional spaces has 3 axes: x, y, z
It is a right-handed system: ie., if you take your right hand point your fingers in the positive x-direction and curl them toward the positive y-direction, your thumb points in the positive z-direction.

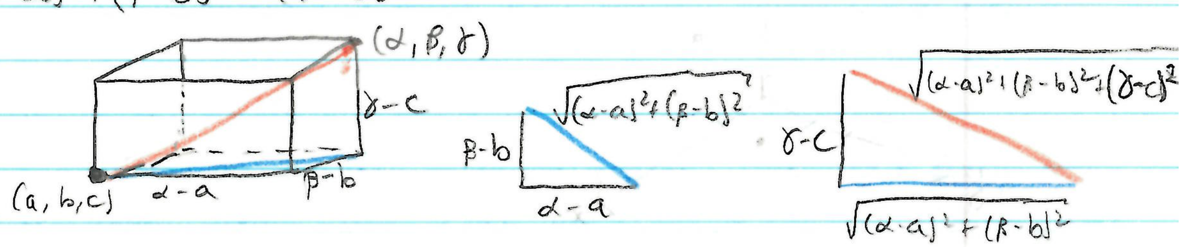
$\mathbb{R}^3$ has 3 coordinate planes:
- xy-plane aka $z=0$ plane
- xz-plane aka $y=0$ plane
- yz-plane aka $x=0$ plane

Distance formula:

In $\mathbb{R}^2$, the distance between (a, b) and (α, β) is given by $\sqrt{(\alpha-a)^2+(\beta-b)^2}$ (by the Pythagorean Theorem)



In $\mathbb{R}^3$, distance between (a,b,c) and (α,β,γ) is $\sqrt{(\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2}$



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A sphere is the set of points (x,y,z) that have the same distance r from the center (a,b,c).

i.e., $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r$

so standard form: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Ex 1. The sphere with center $(1,-3,0)$ and radius 4 has standard form

$$(x-1)^2+(y+3)^2+z^2=16$$

Ex 2. Show that the equation

$$x^2+y^2+z^2-2x+6y+4=0$$

is a sphere and find its center and radius. Describe its intersection with the coordinate planes.

Complete the square:

$$x^2-2x+\underline{1}+y^2+6y+\underline{9}+z^2=-4+\underline{1}+\underline{9}$$

$$(x-1)^2+(y+3)^2+z^2=5$$

center $(1,-3,0)$ and radius $\sqrt{5}$

Intersection with xy-plane is when $z=0$

$$(x-1)^2+(y+3)^2+0^2=5$$

circle $(x-1)^2+(y+3)^2=5,\ z=0$ with center $(1,-3,0)$ and radius $\sqrt{5}$

Intersection with xz-plane is when $y=0$

$(x-1)^2+(3)^2+z^2=5 \Rightarrow (x-1)^2+z^2=-4$ impossible, so does not intersect xz-plane

yz-plane: $x=0 \Rightarrow (-1)^2+(y+3)^2+z^2=5,\ x=0$

circle $(y+3)^2+z^2=4,\ x=0$ with center $(0,-3,0)$ and radius 2

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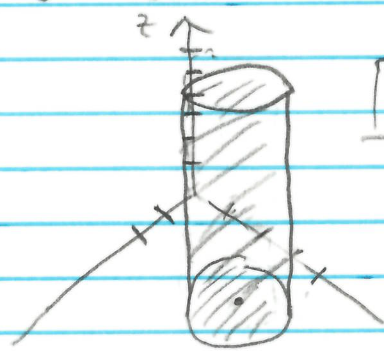
Can use inequalities to describe regions of space.
Any variable not included is allowed to take <u>any</u> value.

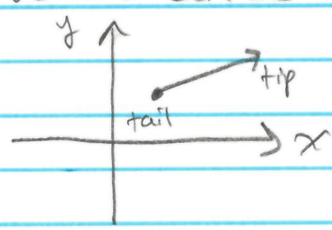
<u>Ex 3</u>. Write inequalities to describe the region <u>inside</u> the right circular cone with base being the circle with center $(2,3,0)$ and radius 1 and of height 6 above the base

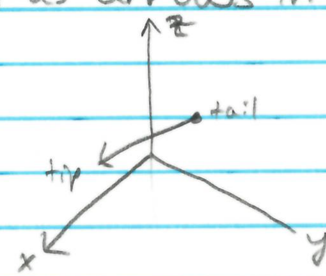


$$\boxed{x^2+y^2<1 \text{ and } 0<z<6}$$

<u>12.2 Vectors</u>

Vectors can be thought of as arrows in space





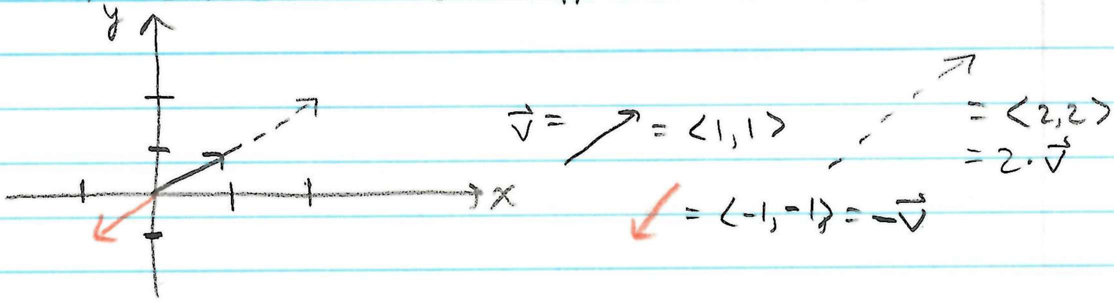
Vectors have magnitude/length and direction.
Vectors can be described in terms of coordinates, so $\langle a,b,c\rangle$ represents the vector with tail at the origin $(0,0,0)$ and tip at (a,b,c).

Therefore, the magnitude/length of the vector $\vec{v}=\langle a,b,c\rangle$ is $|\vec{v}|=\sqrt{a^2+b^2+c^2}$
(and similarly for $\mathbb{R}^2$)

A <u>unit vector</u> is a vector of length 1.

Scalar multiplication.

Given a number p and a vector $\vec{v} = \langle a, b, c \rangle$, the scalar multiplication $p\vec{v} = \langle pa, pb, pc \rangle$ is the vector in the same direction as $\vec{v}$ but whose length is p times that of $\vec{v}$, if $p \geq 0$ (if $p < 0$, $p\vec{v}$ is exactly the opposite direction as $\vec{v}$)



Ex 4. Find a unit vector in the same direction as $\vec{v} = \langle 2, 1, -2 \rangle$

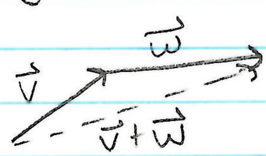
Want length 1, so should scale by $\frac{1}{|\vec{v}|}$

$|\vec{v}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

so $\frac{1}{3}\vec{v} = \frac{1}{3}\langle 2, 1, -2 \rangle = \boxed{\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle}$

Vector Addition.

To add two vectors, place the tail of one at the tip of the other and draw in the vector from the tail which is by itself to the tip which is by itself



component-wise,

if $\vec{v} = \langle a, b, c \rangle$, $\vec{w} = \langle \alpha, \beta, \gamma \rangle$, then

$\vec{v} + \vec{w} = \langle a + \alpha, b + \beta, c + \gamma \rangle$

$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w}$

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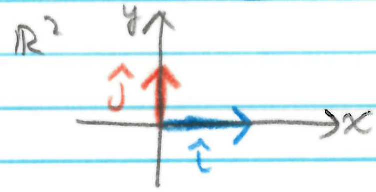
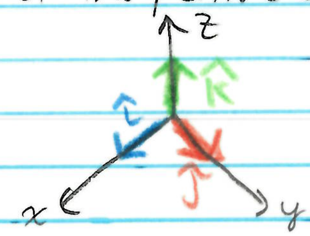
$\mathbb{R}^2$ and $\mathbb{R}^3$ have standard basis vectors which are unit vectors in the directions of the positive axes.





Notice that the vector $\langle a,b,c\rangle$ can also be written as $a\hat{\imath}+b\hat{\jmath}+c\hat{k}$

12.3 The Dot Product

Given vectors $\vec{v}=\langle a,b,c\rangle$ and $\vec{w}=\langle \alpha,\beta,\gamma\rangle$, the dot product $\vec{v}\cdot\vec{w}=a\alpha+b\beta+c\gamma$.
(or, if $\vec{v}=\langle a,b\rangle$, $\vec{w}=\langle \alpha,\beta\rangle$, then $\vec{v}\cdot\vec{w}=a\alpha+b\beta$)

If θ is the angle between $\vec{v}$ and $\vec{w}$, then
$\vec{v}\cdot\vec{w}=|\vec{v}||\vec{w}|\cos\theta$

Ex 5. Find the angle between $\vec{v}=\langle 2,1\rangle$ and $\vec{w}=\langle -1,3\rangle$

$\vec{v}\cdot\vec{w}=(2)(-1)+(1)(3) = -2+3=1$
$|\vec{v}|=\sqrt{4+1}=\sqrt{5}$
$|\vec{w}|=\sqrt{1+9}=\sqrt{10}$

$$\cos\theta=\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}=\frac{1}{\sqrt{5}\sqrt{10}}=\frac{1}{5\sqrt{2}}$$

so $\boxed{\theta=\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right)\approx 81.87^\circ}$

Ex 6. Find two unit vectors which form an angle of $30° = \frac{\pi}{6}$ radians with $\vec{v} = \langle -1, \sqrt{11} \rangle$

Let $\vec{w} = \langle x, y \rangle$. Assume $|\vec{w}| = 1$, so $\sqrt{x^2+y^2} = 1$

or $x^2+y^2 = 1$

Now, $\vec{v} \cdot \vec{w} = (-1)x + \sqrt{11}\,y = -x + \sqrt{11}\,y$

On the other hand $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\frac{\pi}{6}) = (2\sqrt{3})(1)(\frac{\sqrt{3}}{2}) = 3$

since $|\vec{v}| = \sqrt{1+11} = \sqrt{12} = 2\sqrt{3}$ and $|\vec{w}| = 1$

So $3 = -x + \sqrt{11}\,y$ or $x = \sqrt{11}\,y - 3$

Plugging in to $x^2+y^2 = 1$

$(\sqrt{11}\,y - 3)^2 + y^2 = 1 \Rightarrow 11y^2 - 6\sqrt{11}\,y + 9 + y^2 = 1$

$\Rightarrow 12y^2 - 6\sqrt{11}\,y + 8 = 0$

$$y = \frac{6\sqrt{11} \pm \sqrt{396 - 384}}{24} = \frac{6\sqrt{11} \pm 2\sqrt{3}}{24} = \frac{3\sqrt{11} \pm \sqrt{3}}{12}$$

Then $x = \sqrt{11}\,y - 3 = \sqrt{11}\left(\frac{3\sqrt{11} \pm \sqrt{3}}{12}\right) - 3$

so $\boxed{\left\langle \sqrt{11}\left(\frac{3\sqrt{11} \pm \sqrt{3}}{12}\right) - 3,\ \frac{3\sqrt{11} \pm \sqrt{3}}{12} \right\rangle}$

Two vectors are said to be <u>parallel</u> if they are in the same or opposite directions, i.e., if one is a scalar multiple of the other.

e.g., $\langle 2, 1, -3 \rangle$ is parallel to $\langle -6, -3, 9 \rangle$

since $\langle -6, -3, 9 \rangle = -3\langle 2, 1, -3 \rangle$

Two vectors are <u>orthogonal</u> if they are perpendicular, i.e., the angle between them is right, i.e.,
$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(90°) = 0$.
In fact, can only get $\vec{v}\cdot\vec{w}=0$ if $\vec{v}$ and $\vec{w}$ are orthogonal.

$\vec{v}=\langle 1,0,-1\rangle$ and $\vec{w}=\langle 2,7,2\rangle$ are orthogonal since $\vec{v}\cdot\vec{w} = (1)(2)+(0)(7)+(-1)(2) = 2+0+(-2) = 0$

<u>Ex 7</u>. Show that $\langle 2,1,3\rangle$ and $\langle -1,2,2\rangle$ are neither parallel nor orthogonal.

Does $\underbrace{c\langle 2,1,3\rangle}_{\langle 2c,\, c,\, 3c\rangle} = \langle -1,2,2\rangle$ for any c?

no! since this would make $2c=-1$, $c=2$, $3c=2$
$c=-\frac{1}{2}$, $c=2$, $c=\frac{2}{3}$

not the same number!

so not parallel.

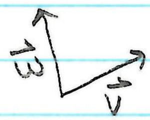
$$\langle 2,1,3\rangle\cdot\langle -1,2,2\rangle = (2)(-1)+(1)(2)+(3)(2)$$
$$= -2+2+6$$
$$= 6 \neq 0$$

so not orthogonal.

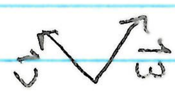
<u>12.4 The Cross Product</u>

Given two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, if $\vec{a}$ and $\vec{b}$ are not parallel, they determine a plane. To get a vector which is orthogonal to that plane (and hence to both $\vec{v}$ and $\vec{w}$) we take the cross product $\vec{v}\times\vec{w}$ or $\vec{w}\times\vec{v}$.

Geometrically speaking, the cross product satisfies the right hand rule, so $\vec{v} \times \vec{w}$ is a vector with magnitude $|\vec{v}||\vec{w}| \sin\theta$ (θ is the smallest angle between $\vec{v}$ and $\vec{w}$) so that if you take your right hand, point your fingers in the direction of $\vec{v}$ and curl them toward $\vec{w}$, your thumb points in the direction of $\vec{v} \times \vec{w}$.



$\vec{v} \times \vec{w}$ is coming out of the board.



$\vec{v} \times \vec{w}$ is going into the board

If $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle \alpha, \beta, \gamma \rangle$, then

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}$$, which is a determinant

Can do $\begin{vmatrix} b & c \\ \beta & \gamma \end{vmatrix} \hat{i} - \begin{vmatrix} a & c \\ \alpha & \gamma \end{vmatrix} \hat{j} + \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \hat{k}$

or $\left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ \alpha & \beta & \gamma \end{matrix}\right| \begin{matrix} \hat{i} & \hat{j} \\ a & b \\ \alpha & \beta \end{matrix}$

multiply the contents of each circle, adding for orange circles and subtracting for blue circles

$$b\gamma\hat{i} + c\alpha\hat{j} + a\beta\hat{k} - b\alpha\hat{k} - c\beta\hat{i} - a\gamma\hat{j}$$
$$= (b\gamma - c\beta)\hat{i} + (c\alpha - a\gamma)\hat{j} + (a\beta - b\alpha)\hat{k}$$

$$(b\gamma - c\beta)\hat{i} - (a\gamma - c\alpha)\hat{j} + (a\beta - b\alpha)\hat{k}$$

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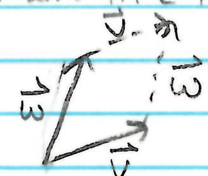
Geometric significance:

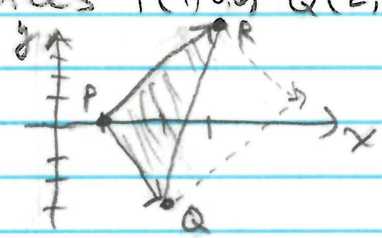
Two non-parallel vectors $\vec{v}$ and $\vec{w}$ determine a parallelogram like this:



The magnitude of the cross product $|\vec{v}\times\vec{w}|$ is equal to the area of the parallelogram (use trig and $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$ to see why)

We can also use the cross product to get the area of a triangle.

Ex 8. Find the area of the triangle with vertices $P(1,0,0)$ $Q(2,-3,0)$, $R(3,4,0)$



Notice the triangle is $\frac{1}{2}$ of the parallelogram determined by $\vec{PQ}$ and $\vec{PR}$,

so $\frac{1}{2}|\vec{PQ}\times\vec{PR}|$

$\vec{PQ} = Q-P = \langle 2-1, -3-0, 0-0\rangle = \langle 1,-3,0\rangle$

$\vec{PR} = R-P = \langle 3-1, 4-0, 0-0\rangle = \langle 2,4,0\rangle$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 0 \\ 2 & 4 & 0 \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 4 & 0 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}\hat{k}$$

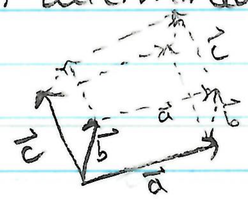
$$= (0-0)\hat{i} - (0-0)\hat{j} + (4-(-6))\hat{k}$$

$$= \langle 0, 0, 10\rangle$$

$|\vec{PQ}\times\vec{PR}| = \sqrt{0+0+10^2} = 10$

so $\frac{1}{2}\cdot 10 = \boxed{5 \text{ units}^2}$

A parallelepiped is a solid with all faces parallelograms and is determined by 3 non-coplanar vectors



The volume of a parallelepiped is equal to
$|\vec{a} \cdot (\vec{b} \times \vec{c})|$ (order of the vectors does not matter).

Ex 9. Find the volume of the parallelepiped determined by $\vec{a} = \langle 1, 0, 2 \rangle$, $\vec{b} = \langle -1, 2, 1 \rangle$, $\vec{c} = \langle 0, -2, 3 \rangle$

$$\vec{b} \times \vec{c} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -1 & 2 & 1 & -1 & 2 \\ 0 & -2 & 3 & 0 & -2 \end{array} \right|$$

$$= 6\hat{i} + 0\hat{j} + 2\hat{k} - 0\hat{k} - (-2)\hat{i} - (-3)\hat{j}$$
$$= 8\hat{i} + 3\hat{j} + 2\hat{k}$$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (1)(8) + (0)(3) + (2)(2) = 8 + 0 + 4 = 12$

$|\vec{a} \cdot (\vec{b} \times \vec{c})| = |12| = 12$

So the parallelepiped has a volume of $\boxed{12 \text{ units}^3}$