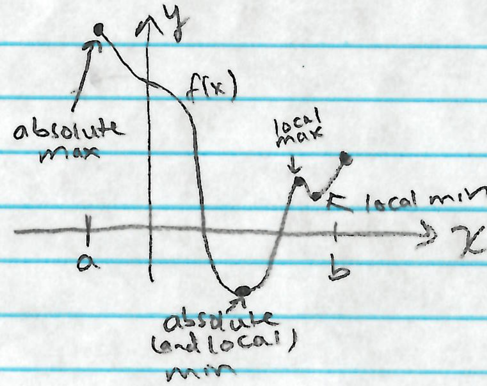
MA 261 - Lesson 10

Constrained Optimization

14.7.

When trying to do optimization in real life, we often have constraints that we are subject to. For example, if we have to make a box, its volume will be $V(l,w,h) = lwh$. Mathematically, V has a domain of all of $\mathbb{R}^3$, but in context, it doesn't make sense for any of the variables to be negative. We may also have constraints on the maximum value of the variables, due to finite resources.

If you rember from calculus I, if you had a continuous function f constrained to a closed interval $[a,b]$, then f obtained absolute extrema somewhere in this interval (Extreme Value Theorem)



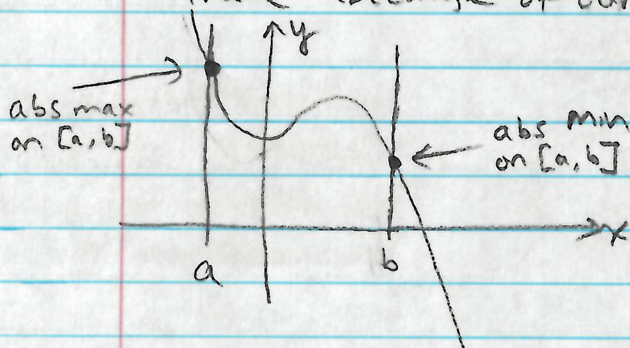
We now look at these concepts in a multivariate setting.

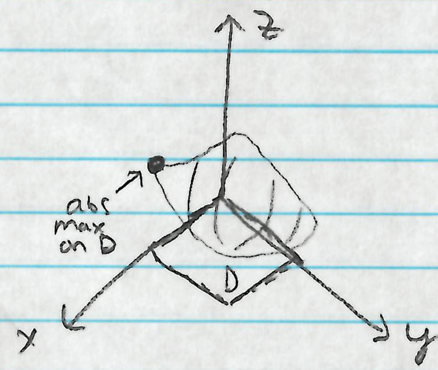
Let f be a continuous function defined on a closed region D of $\mathbb{R}^2$ (all the boundary points of D are in D). f has an absolute max at (a,b) in D if $f(a,b) \geq f(x,y)$ for all (x,y) in D, and f has an absolute min at (a,b) in D if $f(a,b) \leq f(x,y)$ for all (x,y) in D.

Just like for single variable functions, we get

<u>Extreme Value Theorem</u>. Let f be a continuous function defined on a closed region D. Then f attains an absolute minimum value and an absolute maximum value somewhere on D.

Much like with single variable functions, an absolute extremum may occur <u>on the boundary</u> of D, even if f does not have a critical point there (because of our non-mathematical constraints)



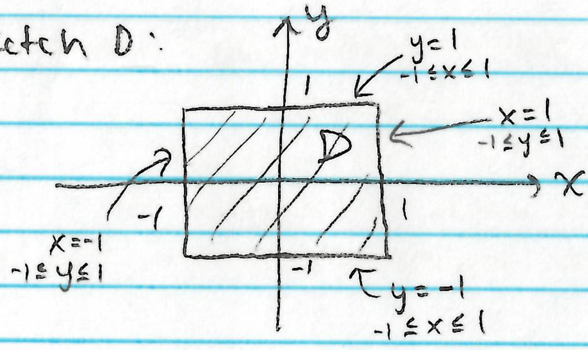


To find absolute extrema of a function which is continuous on a closed region D,

- find all critical points of f occurring <u>in</u> D
- "Parameterize" f with respect to all boundary curves of D and find absolute extrema <u>on</u> the boundary
- Determine abs extrema from all of the above points.

Ex 1. Find the absolute max and min values of $f(x,y) = x^2 + y^2 + x^2y + 4$ on $D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$.

Sketch D:



Find critical points: $f_x = 2x + 2xy \overset{\text{set}}{=} 0$, $f_y = 2y + x^2 \overset{\text{set}}{=} 0$

$$2x(1+y) = 0$$
$$x = 0 \text{ or } y = -1$$

If $x = 0$, then $2y + (0)^2 = 0 \Rightarrow y = 0$ $\quad (0,0)$

If $y = -1$, then $-2 + x^2 = 0 \Rightarrow x = \pm\sqrt{2}$ $\quad (\pm\sqrt{2}, -1)$

$(0,0)$ is in D, but $(\pm\sqrt{2}, -1)$ are outside of D, so we will check $(0,0)$.

$f(0,0) = \boxed{4}$

On the upper line, $y = 1$, so $f(x,y) = f(x,1) = x^2 + (1)^2 + x^2(1) + 4 = 2x^2 + 5$, $-1 \le x \le 1$

This has a critical # of $0, -1, 1$: $f(0,1) = \boxed{5}$, $f(-1,1) = \boxed{7}$, $f(1,1) = \boxed{7}$

On the lower line, $y = -1$, so $f(x,y) = f(x,-1) = x^2 + (-1)^2 + x^2(-1) + 4 = 5$, $-1 \le x \le 1$

This has a constant value of $\boxed{5}$ along the line.

On the right line, $x = 1$, so $f(x,y) = f(1,y) = (1)^2 + y^2 + (1)^2 y + 4 = y^2 + y + 5$, $-1 \le y \le 1$

This has critical #s $-\frac{1}{2}, -1, 1$: $f(1, -\frac{1}{2}) = \boxed{4.75}$, $f(1,-1) = \boxed{6}$, $f(1,1) = \boxed{7}$

On the left line, $x = -1$, so $f(x,y) = f(-1,y) = (-1)^2 + y^2 + (-1)^2 y + 4 = y^2 + y + 5$, $-1 \le y \le 1$

which has the same candidates as the right line.

The absolute min value is 4 at $(0,0)$

absolute max value is 7 at $(1,1)$ and $(-1,1)$

Ex 2. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4,2,0)$.

In general, distance between (x,y,z) and $(4,2,0)$ is $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$, but we are subjecting this to the constraint $z^2 = x^2 + y^2$.

$$d(x,y,z) = \sqrt{(x-4)^2 + (y-2)^2 + (x^2+y^2)}$$
$$= \sqrt{x^2 - 8x + 16 + y^2 - 4y + 4 + x^2 + y^2}$$
$$= \sqrt{2x^2 - 8x + 2y^2 - 4y + 20}$$

Since distance is positive, it is minimized when its square is.

$$d^2 = 2x^2 - 8x + 2y^2 - 4y + 20$$

$d^2_x = 4x - 8 \overset{\text{set}}{=} 0$, $\quad d^2_y = 4y - 4 \overset{\text{set}}{=} 0$

$x = 2$ $\quad\quad y = 1$

$d^2_{xx} = 4$, $d^2_{yy} = 4$, $d^2_{xy} = 0$

$D = 16 > 0$, $d^2_{xx} > 0$ so indeed a minimum

$z^2 = (2)^2 + (1)^2 = 4 + 1 = 5$ so $z = \pm\sqrt{5}$

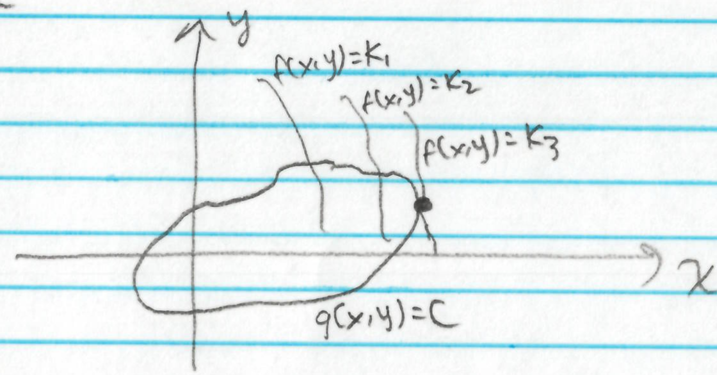
Closest points are $\boxed{(2,1,\sqrt{5}) \text{ and } (2,1,-\sqrt{5})}$

14.8 Lagrange Multipliers

Suppose you want to optimize $f(x,y)$ subject to the constraint $g(x,y) = c$. Sometimes it is not easy to parameterize $f(x,y)$ with respect to the constraint.

Notice that $g(x,y) = C$ is a level curve of $g(x,y)$. To optimize $f(x,y)$ subject to the constraint $g(x,y) = C$, we are trying to find the largest (or smallest) level curve of $f(x,y)$ that intersects the level curve $g(x,y) = C$

e.g.,



Notice that in order to be the largest level curve of $f(x,y)$ intersecting $g(x,y) = C$, the tangent vectors to the level curves must be parallel.

This happens when $\nabla f = \lambda \nabla g$ for some constant λ.

To find the absolute extrema of $f(x,y)$ subject to the constraint $g(x,y) = C$, find all points (a,b) so that $\nabla f(a,b) = \lambda \nabla g(a,b)$ and that satisfy $g(a,b) = C$. Then test which are extreme values by plugging into f.

i.e., solve the system of equations
$f_x = \lambda g_x$, $f_y = \lambda g_y$, $g(x,y) = C$
and test the values.

There is no general process to solve the system! It might require creative thinking.

Ex 3. Use Lagrange multipliers to find the absolute extrema of $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$

$f(x,y) = xy, \quad g(x,y) = 4x^2 + y^2$

$\nabla f = \langle y, x \rangle \overset{set}{=} \lambda \nabla g = \lambda \langle 8x, 2y \rangle = \langle \lambda 8x, \lambda 2y \rangle$

$y = \lambda 8x, \quad x = \lambda 2y, \quad 4x^2 + y^2 = 8$

Notice: multiplying 1st equation by x and 2nd by y,

$xy = \lambda 8x^2, \quad xy = \lambda 2y^2$

So $\lambda 8x^2 = \lambda 2y^2$

Satisfied if $\lambda = 0$ or if $8x^2 = 2y^2 \iff 4x^2 = y^2$

Case 1: $4x^2 = y^2$

Then $4x^2 + y^2 = 8 \iff 4x^2 + 4x^2 = 8 \iff x^2 = 1 \iff x = \pm 1$

If $x = \pm 1$, then $4(\pm 1)^2 = y^2 \iff y = \pm 2$

get points $(-1,-2), (-1,2), (1,-2), (1,2)$

Case 2: $\lambda = 0$: By the first equation $(y = \lambda 8x)$, get $y = 0$

By the second eqn $(x = \lambda 2y)$, get $x = 0$

But $4(0)^2 + (0)^2 = 8$ is false.

No points from this.

$f(-1,-2) = 2, \quad f(-1,2) = -2, \quad f(1,-2) = -2, \quad f(1,2) = 2$

$$\boxed{\begin{array}{l} \text{max value } 2 \text{ at } (-1,-2) \text{ and } (1,2) \\ \text{min value } -2 \text{ at } (-1,2) \text{ and } (1,-2) \end{array}}$$

Same argument applies to any number of variables.

Ex 4. Use Lagrange multipliers to find the absolute extrema of $f(x,y,z) = e^{xyz}$ subject to the constraint $2x^2+y^2+z^2=24$.

$\nabla f = \langle yz e^{xyz}, xz e^{xyz}, xy e^{xyz} \rangle = \lambda \nabla g = \langle \lambda 4x, \lambda 2y, \lambda 2z \rangle$

$yz e^{xyz} = \lambda 4x$, $xz e^{xyz} = \lambda 2y$, $xy e^{xyz} = \lambda 2z$, $2x^2+y^2+z^2=24$

Multiply 1st eqn by x, 2nd by y, 3rd by z

$xyz e^{xyz} = \lambda 4x^2$, $xyz e^{xyz} = \lambda 2y^2$, $xyz e^{xyz} = \lambda 2z^2$

$\lambda 4x^2 = \lambda 2y^2 = \lambda 2z^2$

True if $\lambda = 0$ or if $4x^2 = 2y^2 = 2z^2 \iff 2x^2 = y^2 = z^2$

Case 1. $2x^2 = y^2 = z^2$

Then $2x^2+y^2+z^2 = 24 \iff 2x^2+2x^2+2x^2 = 24 \iff x = \pm 2$

Then $y^2 = z^2 = 2(\pm 2)^2 = 8$ so $y = \pm 2\sqrt{2}$, $z = \pm 2\sqrt{2}$

$(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (all combinations)

Case 2. $\lambda = 0$

From first equation ($yz e^{xyz} = \lambda 4x$), get $y=0$ or $z=0$

From second equation ($xz e^{xyz} = \lambda 2y$) get $x=0$ or $z=0$

From third eqn. ($xy e^{xyz} = \lambda 2z$) get $x=0$ or $y=0$

From fourth eqn. ($2x^2+y^2+z^2=24$), impossible for all three to be 0.

But at least two have to be 0.

Case 2.1. $x=y=0$: $z^2 = 24 \iff z = \pm 2\sqrt{6}$ $(0, 0, \pm 2\sqrt{6})$

Case 2.2. $x=z=0$: $y^2 = 24 \iff y = \pm 2\sqrt{6}$ $(0, \pm 2\sqrt{6}, 0)$

Case 2.3. $y=z=0$: $2x^2 = 24 \iff x = \pm 2\sqrt{3}$ $(\pm 2\sqrt{3}, 0, 0)$

f(case 2 points) = 1

f(case 1 points) = e^{16} or e^{-16} (depending on signs)

max value e^{16} at $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (even # of negatives)

min value e^{-16} at $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (odd # of negatives)

Ex 5. Find the extreme values of $f(x,y) = 2x^2+3y^2-4x-5$ subject to the constraint $x^2+y^2 \le 16$

Boundary of region is $x^2+y^2=16$, use Lagrange multipliers to check. Also check inside region at critical points.

$f_x = 4x-4 \overset{set}{=} 0$, $f_y = 6y \overset{set}{=} 0$

$x=1$, $y=0$ $(1,0)$

$1^2+0^2 \le 16$ ✓ in region

$\nabla f = \langle 4x-4, 6y\rangle = \lambda \nabla g = \langle \lambda 2x, \lambda 2y\rangle$

$4x-4 = \lambda 2x$, $\underbrace{6y = \lambda 2y}$, $x^2+y^2=16$

$6y-2y\lambda = 0$

$2y(3-\lambda)=0$

$y=0$ or $\lambda = 3$

Case 1. $y=0$: $x^2=16 \Leftrightarrow x=\pm 4$ $(\pm 4, 0)$

Case 2. $\lambda = 3$: $4x-4=6x \Leftrightarrow -4=2x \Leftrightarrow x=-2$

$(-2)^2+y^2=16 \Leftrightarrow y^2=12 \Leftrightarrow y=\pm 2\sqrt{3}$ $(-2, \pm 2\sqrt{3})$

$f(1,0) = 2(1)^2+3(0)^2-4(1)-5 = -7$

$f(4,0) = 2(4)^2+3(0)^2-4(4)-5 = 11$

$f(-4,0) = 2(-4)^2+3(0)^2-4(-4)-5 = 43$

$f(-2,2\sqrt{3}) = 2(-2)^2+3(2\sqrt{3})^2-4(-2)-5 = 47$

$f(-2,-2\sqrt{3}) = 47$

max value 47 at $(-2, \pm 2\sqrt{3})$

min value -7 at $(1,0)$