MA 261 - Lesson 11

Double Integrals Over Rectangles (15.1)

How do you find the volume under a surface 
\( z = f(x,y) \) and above the rectangle
\( R = [a,b] \times [c,d] = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d \} \)?

We denote this by \( \iint_R f(x,y) \, dA \), which we will see later why we denote it this way.

If the surface is nice enough, we can find the volume geometrically.

**Ex. 1.** Find \( \iint_R (4-2y) \, dA \), \( R = [0,1] \times [0,1] \)

The surface is \( z = 4-2y \), which when plotted on the \( yz \)-plane is

\[
\begin{align*}
\text{Volume} &= \text{area of region in } yz \text{-plane} \\
&\quad \times \text{distance in } x \text{-direction}
\end{align*}
\]

\[
\text{Area} = \frac{1}{2} (b - a) h = \frac{1}{2} (4 - 2)(1) = 3
\]

\[
\text{distance} = 1
\]

\[
\text{volume} = 3
\]
If the surface isn't nice enough, though, we need another approach. Like approximating the area under a curve in Calc 1, we approximate the volume under the surface by breaking the volume up into rectangular prisms that approximate the volume.

We do this by breaking \( R \) into smaller rectangles, \( m \) in the \( x \)-direction and \( n \) in the \( y \)-direction. We then choose a sample point in each rectangle which we will use to determine the height of the rectangular prism. This is often done by choosing a corner or a midpoint of each rectangle.

\[
R = [0, 3] \times [0, 2] \quad \text{with} \quad m = 3, \ n = 2
\]

with a sample point in each rectangle.

Notice, \( Ax = \frac{b-a}{m} \), \( Ay = \frac{d-c}{n} \) and the area of each rectangle is \( \Delta A = AxAy \) (when \( R = [a,b] \times [c,d] \)).

Doing this, we get a Riemann sum

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]

approximating the volume of the solid.

Where \((x_{ij}^*, y_{ij}^*)\) is the sample point.
Ex 2. If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2$ and $n = 3$ to approximate the value of $SS_R (xy^2 + 1) \, dA$. With sample points:
(a) upper left corners, (b) midpoints

\[
\begin{array}{c|c|c|c|c}
& 1 & 2 & 3 & 4 \\
\hline
0 & - & - & - & - \\
-1 & - & - & - & - \\
\hline
0 & - & - & - & - \\
2 & - & - & - & - \\
\hline
4 & - & - & - & - \\
\end{array}
\]

$\Delta x = \frac{4 - 0}{2} = 2$
$\Delta y = \frac{2 - (-1)}{3} = 1$
So $\Delta A = (2)(1) = 2$

(a) upper left corners are $(0,0), (0,1), (0,2), (2,0), (2,1), (2,2)$
So $V \approx \left( f(0,0) + f(0,1) + f(0,2) + f(2,0) + f(2,1) + f(2,2) \right) \Delta A$
\[= \left( 0 + 1 + 1 + 1 + 3 + 9 \right)(2)\]
\[= (16)(2) = 32\]
(Since $f(x,y) = xy^2 + 1$)

(b) midpoints are $(1,\frac{1}{2}), (3,\frac{3}{4}), (1,\frac{3}{2}), (3,\frac{3}{2}), (1,\frac{3}{2}), (3,\frac{3}{2})$
So $V \approx \left( f(1,\frac{1}{2}) + f(3,\frac{3}{4}) + f(1,\frac{3}{2}) + f(3,\frac{3}{2}) + f(1,\frac{3}{2}) + f(3,\frac{3}{2}) \right) \Delta A$
\[= \left( \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} \right) 2\]
\[= (17)(2) = 34\]

Our approximation gets better as $\Delta A \to 0$
(i.e., as $n \to \infty$ and $m \to \infty$)

In fact, $SS_R f(x,y) \, dA = \lim_{m,n \to \infty} \sum \sum f(x_{ij}, y_{ij}) \Delta A$

But how do you calculate a double integral?
Partial Integrals and Iterated Integrals

Suppose you have \( f(x,y) \) and want to integrate it just with respect to the variable \( x \). We can by treating \( y \) as a constant. Our final result will be a function depending only on \( y \). Similarly for the other variable. This is a partial integral.

Ex 3. Compute the partial integrals

\[
\int_0^1 x^2 \sqrt{y+3} \, dx \quad \text{and} \quad \int_0^1 x^2 \sqrt{y+3} \, dy
\]

\[
\int_0^1 x^2 \sqrt{y+3} \, dx = \frac{1}{3} x^3 \sqrt{y+3} \bigg|_{x=0}^{x=1} = \frac{1}{3} \sqrt{y+3}
\]

\[
\int_0^1 x^2 (y+3)^{1/2} \, dy = \frac{2}{3} x^2 (y+3)^{3/2} \bigg|_{y=0}^{y=1} = \frac{2 x^2}{3} \left( 8 - 3 \sqrt{3} \right)
\]

We can then compute iterated integrals

\[
\int_0^1 \left[ \int_0^1 f(x,y) \, dx \right] \, dy , \quad \text{doing the inside integral first.}
\]

Ex 4. Calculate the iterated integral

\[
\int_0^1 \int_0^3 \frac{\ln y}{xy} \, dy \, dx
\]

Let \( u = \ln y \), then \( du = \frac{1}{y} \, dy \)

\[
\int_0^1 \int_0^{\ln(5)} \frac{u}{x} \, du \, dx = \int_0^1 \left( \frac{1}{2x} u^2 \bigg|_{u=0}^{u=\ln(5)} \right) \, dx
\]

\[
= \int_0^1 \frac{\ln^2(5)}{2} \, x \, dx
\]

\[
= \frac{\ln^2(5)}{4} x^2 \bigg|_0^1 = \frac{\ln^2(5)}{4} - \frac{\ln^2(5)}{4} = 0
\]
Suppose you have a surface \( z = f(x, y) \) over a rectangle \( R = [a, b] \times [c, d] \).

For a fixed value of \( y \), \( \int_a^b f(x, y) \, dx \) gives you the area of the cross-section of the solid by slicing it with that plane. Then integrating along all \( y \)-values from \( c \) to \( d \), we get the entire volume.

So \( \iiint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy \).

We could do a similar argument to show we could switch the order of the variables.

**Fubini's Theorem.** If \( z = f(x, y) \) is a continuous function on the rectangle \( R = [a, b] \times [c, d] \), then
\[ \iiint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx. \]

This lesson only deals with rectangular regions. The argument to establish the connection between double and iterated integrals will still work for non-rectangular regions, but it won't look as clean and nice as Fubini's Theorem.
Ex 5. Compute the double integral 
\[ \iint_{R} (y + xy^2) \, dx \, dy \]
where \( R = \{(xy) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\} \). 

Easier to integrate with respect to \( x \), so by Fubini's Theorem,
\[ \int_{1}^{2} \int_{0}^{x} (y + xy^2) \, dy \, dx \]
\[ = \int_{1}^{2} \left[ xy + \frac{1}{2}x^2y^2 \right]_{y=1}^{y=2} \, dx \]
\[ = \int_{1}^{2} (2y + 2y^2) \, dx \]
\[ = y^2 + 2y^2 \bigg|_{y=1}^{y=2} \]
\[ = 4 - 1 - (1 - 2) = 3 - (-1) = 4 \]

Ex 6. Find the volume under the plane 
\[ 2x + 3y - z = 4 \]
above the rectangle \( R = [0, 1] \times [1, 2] \).

\[ z = 2x + 3y - 4 \]

\[ \iiint_{R} (2x + 3y - 4) \, dx \, dy \]
\[ = \int_{1}^{2} \int_{0}^{x} (2x + 3y - 4) \, dy \, dx \]
\[ = \int_{1}^{2} \left[ 2xy + \frac{3}{2}xy^2 - 4y \right]_{y=0}^{y=2} \, dx \]
\[ = \int_{1}^{2} (2x + 3y - 4) \, dx \]
\[ = \int_{1}^{2} (3y - 3) \, dx \]
\[ = \frac{3}{2}y^2 - 3y \bigg|_{y=1}^{y=2} \]
\[ = (6 - 6) - \left( \frac{3}{2} - 3 \right) \]
\[ = \frac{3}{2} \]