MA 261 - Lesson 11

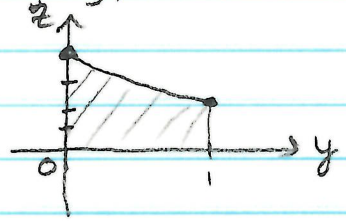
Double Integrals Over Rectangles (15.1)

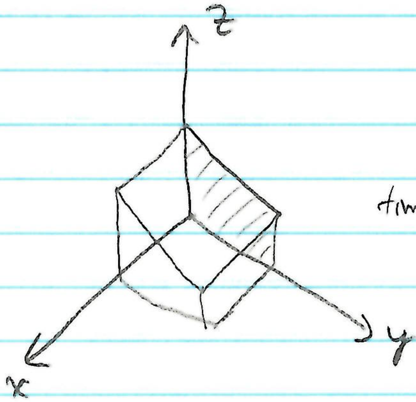
How do you find the volume under a surface $z=f(x,y)$ and above the rectangle
$R=[a,b]\times[c,d]=\{(x,y)\mid a\le x\le b,\ c\le y\le d\}$?
We denote this by $\iint_R f(x,y)\,dA$, which we will see later why we denote it this way.

If the surface is nice enough, we can find the volume geometrically.

Ex 1. Find $\iint_R (4-2y)\,dA$, $R=[0,1]\times[0,1]$

The surface is $z=4-2y$, which when plotted on the yz-plane is





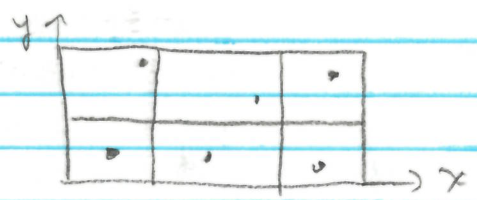
volume is area of region in yz-plane times distance in x-direction

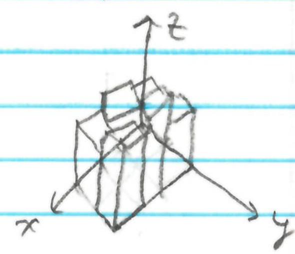
$$\text{Area}=\tfrac{1}{2}(b_1+b_2)h=\tfrac{1}{2}(4+2)(1)=3$$
$$\text{distance}=1$$

volume is $\boxed{3}$

If the surface isn't nice enough, though, we need another approach. Like approximating the area under a curve in Calc 1, we approximate the volume under the surface by breaking the volume up into rectangular prisms that approximate the volume.

We do this by breaking R into smaller rectangles, m in the x-direction and n in the y-direction. We then choose a sample point in each rectangle which we will use to determine the height of the rectangular prism. This is often done by choosing a corner or a midpoint of each rectangle.





$R = [0,3] \times [0,2]$ with $m=3$, $n=2$
with a sample point in each rectangle

Notice, $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$ and the area of each rectangle is $\Delta A = \Delta x \Delta y$
(when $R = [a,b] \times [c,d]$)

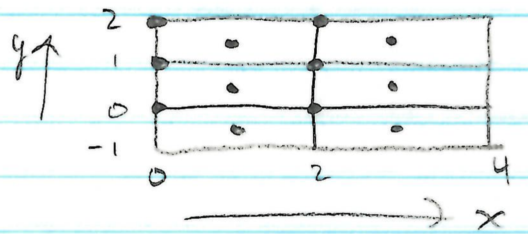
Doing this, we get a Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

approximating the volume of the solid

where (x_{ij}^*, y_{ij}^*) is the sample point.

Ex 2. If $R = [0,4] \times [-1,2]$, use a Riemann sum with $m=2$ and $n=3$ to approximate the value of $\iint_R (xy^2+1)\, dA$ with sample points
(a) upper left corners, (b) midpoints



$\Delta x = \frac{4-0}{2} = 2$

$\Delta y = \frac{2-(-1)}{3} = 1$

so $\Delta A = (2)(1) = 2$

(a) upper left corners are $(0,0), (0,1), (0,2), (2,0), (2,1), (2,2)$

So $V \approx \left(f(0,0) + f(0,1) + f(0,2) + f(2,0) + f(2,1) + f(2,2) \right) \Delta A$

$= (1 + 1 + 1 + 1 + 3 + 9)(2)$

$= (16)(2) = \boxed{32}$

(since $f(x,y) = xy^2+1$)

(b) midpoints are $(1,-\frac{1}{2}), (3,-\frac{1}{2}), (1,\frac{1}{2}), (3,\frac{1}{2}), (1,\frac{3}{2}), (3,\frac{3}{2})$

So $V \approx \left(f(1,-\frac{1}{2}) + f(3,-\frac{1}{2}) + f(1,\frac{1}{2}) + f(3,\frac{1}{2}) + f(1,\frac{3}{2}) + f(3,\frac{3}{2}) \right) \Delta A$

$= \left(\frac{5}{4} + \frac{7}{4} + \frac{5}{4} + \frac{7}{4} + \frac{13}{4} + \frac{31}{4} \right) 2$

$= (17)(2) = \boxed{34}$

Our approximation gets better as $\Delta A \to 0$
(i.e., as $n \to \infty$ and $m \to \infty$)

In fact, $\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$

But how do you calculate a double integral?

Partial Integrals and Iterated Integrals

Suppose you have $f(x,y)$ and want to integrate it just with respect to the variable x. We can by treating y as a constant. Our final result will be a function depending only on y. Similarly for the other variable. This is a partial integral.

Ex 3. Compute the partial integrals

$$\int_0^1 x^2\sqrt{y+3}\,dx \quad \text{and} \quad \int_0^1 x^2\sqrt{y+3}\,dy$$

$$\int_0^1 x^2\sqrt{y+3}\,dx = \frac{1}{3}x^3\sqrt{y+3}\Big|_{x=0}^{x=1} = \boxed{\frac{1}{3}\sqrt{y+3}}$$

$$\int_0^1 x^2(y+3)^{1/2}\,dy = \frac{2x^2}{3}(y+3)^{3/2}\Big|_{y=0}^{y=1} = \boxed{\frac{2x^2}{3}(8-3\sqrt{3})}$$

We can then compute iterated integrals

$$\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$, doing the inside integral first.

Ex 4. Calculate the iterated integral

$$\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx$$

Let $u = \ln y$, then $du = \frac{1}{y}\,dy$

$$\int_1^3\int_0^{\ln(5)} \frac{u}{x}\,du\,dx = \int_1^3\left(\frac{1}{2x}u^2\Big|_{u=0}^{u=\ln(5)}\right)dx$$

$$= \int_1^3 \frac{\ln^2(5)}{2}x\,dx$$

$$= \frac{\ln^2(5)}{4}x^2\Big|_1^3$$

$$= \boxed{2\ln^2(5) - \frac{\ln^2(5)}{4}}$$

Suppose you have a surface $z = f(x,y)$ over a rectangle $R = [a,b] \times [c,d]$

For a fixed value of y, $\int_a^b f(x,y)\,dx$
gives you the area of the cross-section of the solid by slicing it with that plane.
Then integrating along all y-values from c to d, we get the entire volume.

So $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$

We could do a similar argument to show we could switch the order of the variables.

<u>Fubini's Theorem</u>. If $z = f(x,y)$ is a continuous function on the rectangle $R = [a,b] \times [c,d]$,
then $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

This lesson only deals with rectangular regions. The argument to establish the connection between double and iterated integrals will still work for non-rectangular regions, but it won't look as clean and nice as Fubini's Theorem.

Ex 5. Compute the double integral
$\iint_R (y + xy^{-2})\, dA$ where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$

Easier to integrate with respect to x, so by Fubini's Theorem,

$$\int_1^2 \int_0^2 (y + xy^{-2})\, dx\, dy$$

$$= \int_1^2 \left(xy + \tfrac{1}{2}x^2 y^{-2}\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_1^2 (2y + 2y^{-2})\, dy$$

$$= y^2 - 2y^{-1} \Big|_1^2$$

$$= (4-1) - (1-2) = 3 - (-1) = \boxed{4}$$

Ex 6. Find the volume under the plane
$2x + 3y - z = 4$ above the rectangle $R = [0,1] \times [1,2]$

$z = 2x + 3y - 4$

So $\int_1^2 \int_0^1 (2x + 3y - 4)\, dx\, dy = \int_1^2 (x^2 + 3xy - 4x)\Big|_{x=0}^{x=1} dy$

$$= \int_1^2 (1 + 3y - 4)\, dy$$

$$= \int_1^2 (3y - 3)\, dy$$

$$= \tfrac{3}{2}y^2 - 3y \Big|_1^2$$

$$= (6 - 6) - \left(\tfrac{3}{2} - 3\right)$$

$$= \boxed{\tfrac{3}{2}}$$