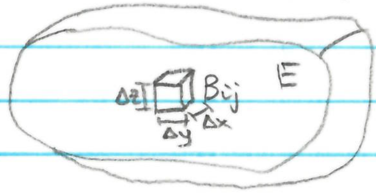
MA 261 - Lesson 14

Triple Integrals (15.6)

You can define a triple integral for a function of 3 variables x, y, and z over some region E in $\mathbb{R}^3$.



For similar reasons, we get

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\,\Delta V$$

For the same reason as Fubini's Theorem (applied to 3 dimensions), a triple integral can be evaluated as iterated integrals in 3 variables.

Now, the inner integral goes from surface to surface, the middle integral goes from curve to curve on the projection of E onto the first-variable = 0 plane, and the outer integral goes from value to value on the same projection.

e.g., for $dz\,dy\,dx$,

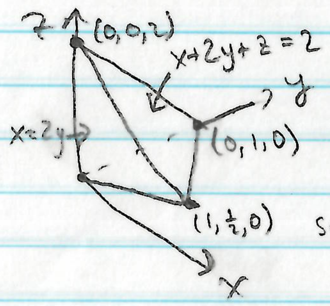
z varies from surface $z = f(x,y)$ bounding E below
to surface $z = g(x,y)$ bounding E above

Now project E onto the xy-plane to get the region D.

y varies from curve $y = h(x)$ bounding D below
to curve $y = j(x)$ bounding D above.

x varies from the minimal x-value in D to the maximal x-value in D.

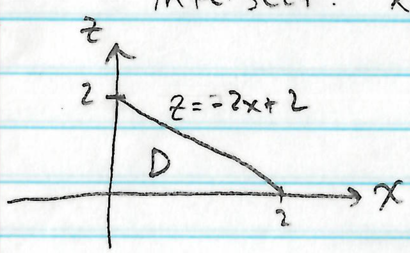
Ex 1. Write the triple integral $\iiint_E f(x,y,z)\,dV$ in all six orders of integration, where E is the tetrahedron bounded by the planes, $x=0$, $z=0$, $x=2y$ and $x+2y+z=2$



since $x+2y+z=2$ and $x=2y$ intersect when $2y+2y+z=2$ which intersects the xy-plane when $4y+0=2 \Rightarrow y=\frac{1}{2}$ and $x=2y=2(\frac{1}{2})=1$

dy first: y varries from surface $y=\frac{1}{2}x$ $(x=2y)$ to $y=1-\frac{1}{2}x-\frac{1}{2}z$ $(x+2y+z=2)$

projection onto xz-plane: notice line of intersection of the two planes is when $x=2y$ and $x+2y+z=2$ intersect: $x+x+z=2 \Longrightarrow 2x+z=2$ or $z=-2x+2$ or $x=-\frac{1}{2}z+1$



dz dx: z goes from curve $z=0$ to curve $z=-2x+2$
x has min value 0 and max value 2

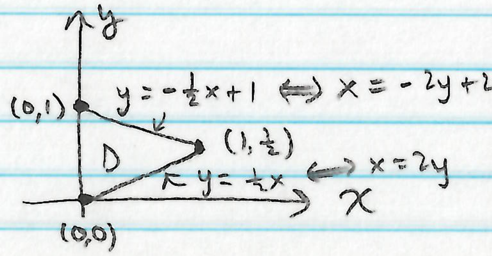
$$\boxed{\int_0^2 \int_0^{-2x+2} \int_{\frac{1}{2}x}^{1-\frac{1}{2}x-\frac{1}{2}z} f(x,y,z)\,dy\,dz\,dx}$$

dx dz: x goes from curve $x=0$ to curve $x=-\frac{1}{2}z+1$
z has min value 0 and max value 2

$$\boxed{\int_0^2 \int_0^{-\frac{1}{2}z+1} \int_{\frac{1}{2}x}^{1-\frac{1}{2}x-\frac{1}{2}z} f(x,y,z)\,dy\,dx\,dz}$$

__dz first__. z goes from surface $z=0$ to surface $z = 2-x-2y$ $(x+2y+z=2)$

Projection onto xy-plane:



__dy dx__: y varies from curve $y=\frac{1}{2}x$ to curve $y=-\frac{1}{2}x+1$
x has min value 0 and max value 1

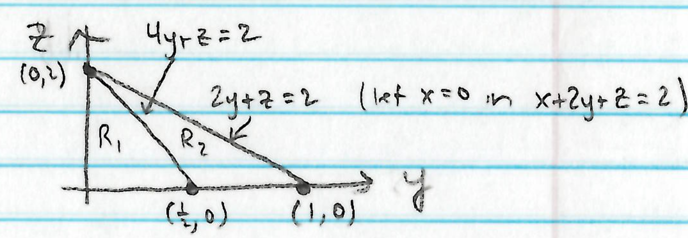
$$\boxed{\int_0^1 \int_{\frac{1}{2}x}^{-\frac{1}{2}x+1} \int_0^{2-x-2y} f(x,y,z)\, dz\, dy\, dx}$$

__dx dy__: x varies from curve $x=0$ to curve $x=2y$ when $0 \le y \le \frac{1}{2}$, x varies from curve $x=0$ to $x=-2y+2$ when $\frac{1}{2} \le y \le 1$

$$\boxed{\int_0^{\frac{1}{2}} \int_0^{2y} \int_0^{2-x-2y} f(x,y,z)\, dz\, dx\, dy + \int_{\frac{1}{2}}^{1} \int_0^{-2y+2} \int_0^{2-x-2y} f(x,y,z)\, dz\, dx\, dy}$$

__dx first__. x goes from surface $x=0$ to surface $x=2y$ (over region 1), from surface $x=0$ to surface $x=2-2y-z$ (over region 2)

Projection onto yz-plane:
planes intersect at $4y+z=2$



dz dy: in R_1, z varies from curve $z=0$ to curve $z = 2-4y$

y has min value 0, max value $\frac{1}{2}$

in R_2, z varies from curve $z = 2-4y$ to curve $z = 2-2y$ for $0 \le y \le \frac{1}{2}$,

from curve $z = 0$ to curve $z = 2-2y$ for $\frac{1}{2} \le y \le 1$

$$\boxed{\underbrace{\int_0^{1/2}\int_0^{2-4y}\int_0^{2y} f(x,y,z)\,dx\,dz\,dy}_{R_1} + \underbrace{\int_0^{\frac{1}{2}}\int_{2-4y}^{2-2y}\int_0^{2-2y-z} f(x,y,z)\,dx\,dz\,dy + \int_{\frac{1}{2}}^{1}\int_0^{2-2y}\int_0^{2-2y-z} f(x,y,z)\,dx\,dz\,dy}_{R_2}}$$

dy dz: in R_1, y varies from curve $y=0$ to $y = \frac{1}{2} - \frac{1}{4}z$

z has min value 0, max value 2

in R_2, y varies from curve $y = \frac{1}{2} - \frac{1}{4}z$ to curve $y = 1 - \frac{1}{2}z$

z has min value 0, max value 2

$$\boxed{\int_0^2\int_0^{\frac{1}{2}-\frac{1}{4}z}\int_0^{2y} f(x,y,z)\,dx\,dy\,dz + \int_0^2\int_{1-\frac{1}{2}z}^{\frac{1}{2}-\frac{1}{4}z}\int_0^{2-2y-z} f(x,y,z)\,dx\,dy\,dz}$$

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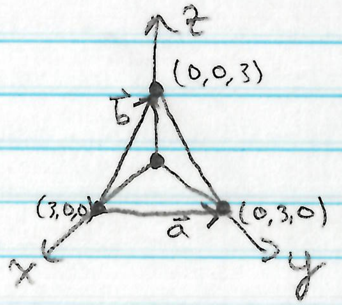
<u>Ex 2</u>. Calculate $\iiint_T x \, dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$.



not hard to do $dz\,dy\,dx$ but need equation of plane.

need normal vector

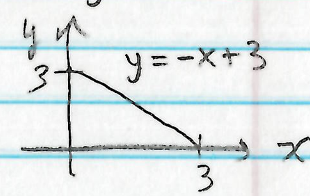
$\vec{a} = \langle 0-3, 3-0, 0-0 \rangle$ and $\vec{b} = \langle 0-3, 0-0, 3-0 \rangle$ are in the plane.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 3 & 0 \\ -3 & 0 & 3 \end{vmatrix} = \langle 9, 9, 9 \rangle$$

using point $(3,0,0)$, get $9(x-3) + 9y + 9z = 0$

or $z = 3 - x - y$

projection onto xy-plane:



$$\int_0^3 \int_0^{-x+3} \int_0^{3-x-y} x \, dz\,dy\,dx$$

$$= \int_0^3 \int_0^{-x+3} xz \Big|_{z=0}^{z=3-x-y} dy\,dx$$

$$= \int_0^3 \int_0^{-x+3} (3x - x^2 - xy)\, dy\,dx$$

$$= \int_0^3 \left(3xy - x^2y - \tfrac{1}{2}xy^2\right)\Big|_{y=0}^{y=-x+3} dx$$

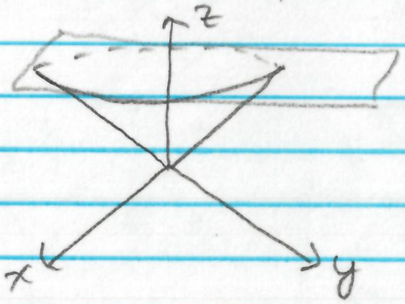
$$= \int_0^3 \left(-3x^2 + 9x + x^3 - 3x^2 - \tfrac{1}{2}x^3 + 3x^2 - \tfrac{9}{2}x\right) dx$$

$$= \int_0^3 \left(\tfrac{1}{2}x^3 - 3x^2 + \tfrac{9}{2}x\right) dx$$

$$= \tfrac{1}{8}x^4 - x^3 + \tfrac{9}{4}x^2 \Big|_0^3$$

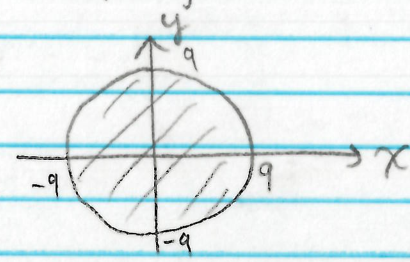
$$= \frac{81}{8} - 27 + \frac{81}{4} = \boxed{\frac{27}{8}}$$

Ex 3. Write $\iiint_E f(x,y,z)\,dV$ as an iterated integral, where E is the region above the cone $z=\sqrt{x^2+y^2}$ and below the plane $z=9$.



projection onto xy-plane is the intersection of the cone with the plane:

$9=\sqrt{x^2+y^2} \Rightarrow 81 = x^2+y^2$



z varies from the surface $z=\sqrt{x^2+y^2}$ to $z=9$

y varies from curve $y=-\sqrt{81-x^2}$ to $y=\sqrt{81-x^2}$

x varies from -9 to 9

$$\int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{\sqrt{x^2+y^2}}^{9} f(x,y,z)\,dz\,dy\,dx$$

Converting the remaining double integral to polar after integrating with respect to z will make it easier.

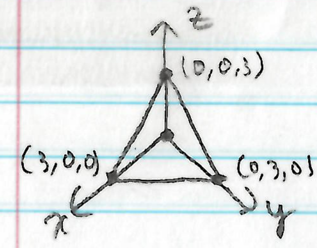
Volumes using Triple Integrals

We saw that we could compute the area of a region D in the xy-plane by taking $\iint_D 1 \, dA$.

Similarly, you can compute the volume of a region E in $\mathbb{R}^3$ by taking

$$\iiint_E 1 \, dV$$

Ex 4. Compute the volume of the tetrahedron from Ex 2.



$$\int_0^3 \int_0^{-x+3} \int_0^{3-x-y} 1 \, dz\, dy\, dx$$

$$= \int_0^3 \int_0^{-x+3} z \Big|_0^{3-x-y} dy\, dx$$

$$= \int_0^3 \int_0^{-x+3} (3-x-y)\, dy\, dx$$

$$= \int_0^3 \left(3y - xy - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=-x+3} dx$$

$$= \int_0^3 \left(-3x + 9 + x^2 - 3x - \tfrac{1}{2}x^2 + 3x - \tfrac{9}{2}\right) dx$$

$$= \int_0^3 \left(\tfrac{1}{2}x^2 - 3x + \tfrac{9}{2}\right) dx$$

$$= \tfrac{1}{6}x^3 - \tfrac{3}{2}x^2 + \tfrac{9}{2}x \Big|_0^3$$

$$= \tfrac{9}{2} - \tfrac{27}{2} + \tfrac{27}{2}$$

$$= \boxed{\tfrac{9}{2}}$$