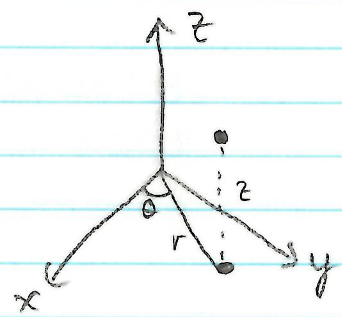
MA 261 – Lesson 15

Triple Integrals in Cylindrical Coordinates (15.7)

In Lesson 14, Ex 3, we saw a triple integral that seemed like we should have dz for part of it but be in polar for the rest. This leads us to cylindrical coordinates.

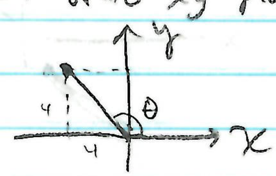
A point (x, y, z) in $\mathbb{R}^3$ (rectangular coordinates) can be given cylindrical coordinates (r, θ, z) where r is the distance of the projection of the point onto the xy-plane from the origin and θ is the angle that projection makes with the positive x-axis



Just like with polar, $r^2 = x^2 + y^2$, so $r = \sqrt{x^2+y^2}$, and $x = r\cos\theta$, $y = r\sin\theta$

Ex 1. Convert $(-4, 4, 4)$ from rectangular to cylindrical coordinates.

z stays as 4, project onto xy-plane:



$r = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{4}{-4}\right) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$
(QII)

$\boxed{\left(4\sqrt{2}, \frac{3\pi}{4}, 4\right)}$

Ex 2. Convert the equation $x^2+y^2+z^2+x+y+z=3$ into cylindrical coordinates.

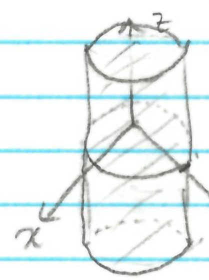
$$\underbrace{x^2+y^2}_{r^2} + z^2 + \underbrace{x} + \underbrace{y} + z = 3$$

$$\boxed{r^2 + r\cos\theta + r\sin\theta + z^2 + z = 3}$$

Ex 3. Sketch the solids

(a) $r \le 3$, (b) $r \le z \le 4$, $0 \le \theta \le \pi$, (c) $r^2 \le z \le 4$

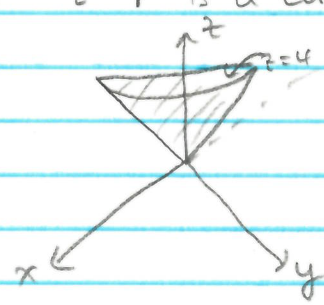
(a)



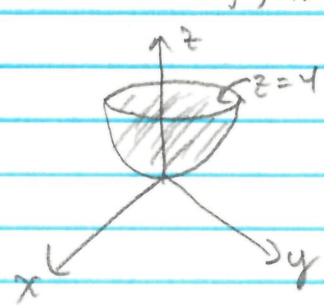
Since θ and z can be anything

(this is why it is "cylindrical" coordinates!)

(b) $z = r$ is a cone ($r = \sqrt{x^2+y^2}$)



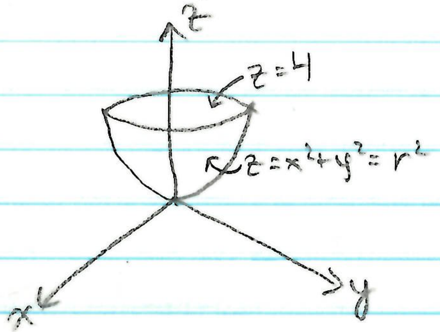
(c) $z = r^2 (= x^2+y^2)$ is a paraboloid opening upward



Just like before, $dV = \underline{\underline{r}}\, dz\, dr\, d\theta$

So converting to cylindrical coordinates can make integrals easier if E is "cylindrical."

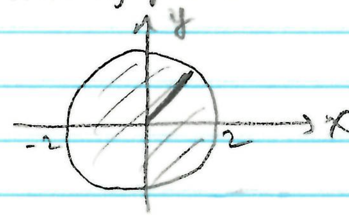
Ex 4. Evaluate $\iiint_E z\, dV$, where E is enclosed by the paraboloid $z = x^2+y^2$ and the plane $z=4$.



projection onto xy-plane:

$r^2 = 4$

$r = 2$



$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}z^2 r \Big|_{z=r^2}^{z=4} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(8r - \frac{1}{2}r^5\right) dr\, d\theta$$

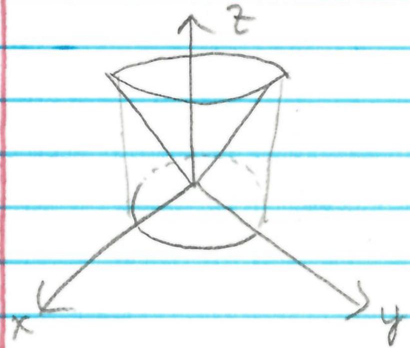
$$= \int_0^{2\pi}\left(4r^2 - \frac{1}{12}r^6\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi}\left(16 - \frac{16}{3}\right) d\theta$$

$$= 2\pi\left(16 - \frac{16}{3}\right)$$

$$= \boxed{\frac{64}{3}\pi}$$

Ex 5. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$



$z^2 = 4x^2+4y^2 = 4r^2$

$z = \pm 2r$, but only need top half

$$\int_0^{2\pi}\int_0^1\int_0^{2r} (r^2\cos^2\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, z \Big|_{z=0}^{z=2r} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{5} r^5 \cos^2\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{2}{5}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{2}{5}\int_0^{2\pi} \frac{1}{2}(1+\cos(2\theta))\, d\theta$$

$$= \frac{1}{5}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi}$$

$$= \frac{1}{5}\left[2\pi + 0 - 0 - 0\right]$$

$$= \boxed{\frac{2}{5}\pi}$$

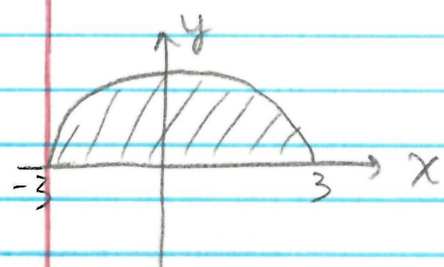
Ex 6. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$

by changing to cylindrical coordinates.

$\sqrt{x^2+y^2} = r, \quad z = 9-x^2-y^2 = 9-r^2$

in the xy-plane, y goes from $y=0$ to $y=\sqrt{9-x^2}$
(top half of $x^2+y^2=9$)

x goes from $x=-3$ to $x=3$



$$\int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r\cdot r \, dz\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{3} r^2 z\Big|_{z=0}^{z=9-r^2} dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{3} (9r^2 - r^4)\, dr\, d\theta$$

$$= \int_0^{\pi} \left(3r^3 - \tfrac{1}{5}r^5\right)\Big|_{r=0}^{r=3} d\theta$$

$$= \int_0^{\pi} \left(81 - \frac{243}{5}\right) d\theta$$

$$= \frac{162}{5}\int_0^{\pi} d\theta$$

$$= \boxed{\frac{162}{5}\pi}$$