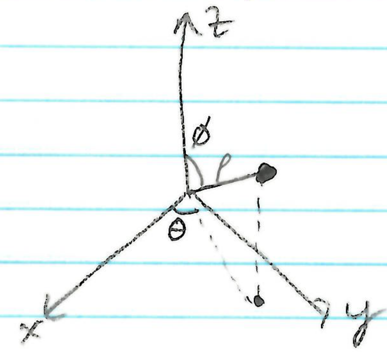
MA 261 – Lesson 16

Triple Integrals Using Spherical coordinates (15.8)

Just like cylindrical coordinates helped with cylindrical regions E in space, spherical coordinates can help with spherical regions of space.

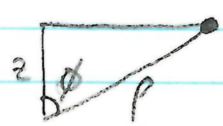
A point (x,y,z) in $\mathbb{R}^3$ can be represented in spherical coordinates (ρ, θ, ϕ), where ρ is the distance between the point and the origin, θ is the angle the point makes with the positive x-axis (project to xy-plane), and ϕ is the angle the point makes with the positive z-axis.



Notice: we only need $0 \leq \phi \leq \pi$ to get all of space

So $\rho = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (adjusted to be in the correct quadrant),

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$



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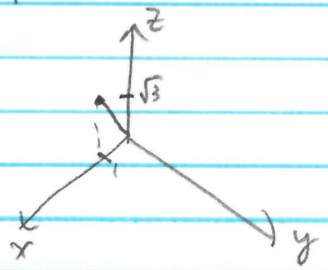
<u>Ex 1</u>. Convert $(1, 0, \sqrt{3})$ from rectangular to spherical coordinates.



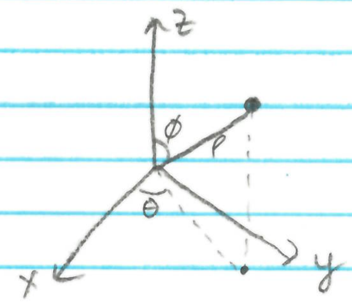
$$\rho = \sqrt{1^2 + 0^2 + \sqrt{3}^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\tfrac{0}{1}\right) = 0$$

$$\phi = \cos^{-1}\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{\pi}{6}$$

$$\boxed{\left(2, 0, \tfrac{\pi}{6}\right)}$$

Given a point (ρ, θ, ϕ) in spherical coordinates, we can covert back to rectangular

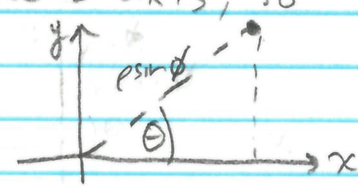


Notice: $z = \rho \cos\phi$ (since $\cos\phi = \frac{z}{\rho}$)

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

(since $\rho \sin\phi$ is the distance away from the z-axis, so



)

<u>Ex 2</u>. Convert $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$ from spherical to rectangular coordinates.

$$x = \rho\cos\theta\sin\phi = 2\cos\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{4}\right) = 2 \cdot 0 \cdot \tfrac{\sqrt{2}}{2} = 0$$

$$y = \rho\sin\theta\sin\phi = 2\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{4}\right) = 2 \cdot 1 \cdot \tfrac{\sqrt{2}}{2} = \sqrt{2}$$

$$z = \rho\cos\phi = 2\cos\left(\tfrac{\pi}{4}\right) = 2\tfrac{\sqrt{2}}{2} = \sqrt{2}$$

$$\boxed{(0, \sqrt{2}, \sqrt{2})}$$

We can use those equations to convert equations into spherical coordinates.

Ex 3. Write the equation in spherical coordinates
$z = x^2 + y^2$

$\rho\cos\phi = (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2$

$\rho\cos\phi = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$

$\cos\phi = \rho\sin^2\phi(\cos^2\theta + \sin^2\theta)$

$\cos\phi = \rho\sin^2\phi$

$\boxed{\rho = \cot\phi\csc\phi}$

Ex 4. (a) Find inequalities to describe a hollow ball with diameter 30 cm and thickness 0.7 cm. Explain how you have positioned the coordinate system.

Center the sphere at the origin. The outer radius is 15 cm and inner radius is 14.3 cm.

$14.3 \le \rho \le 15, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

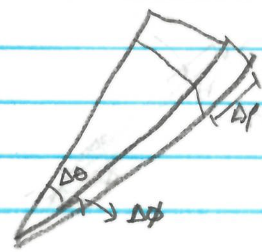
(b) Write inequalities to describe a half of the ball

Could take top half: $14.3 \le \rho \le 15, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \frac{\pi}{2}$

or bottom half: $14.3 \le \rho \le 15, \ 0 \le \theta \le 2\pi, \ \frac{\pi}{2} \le \phi \le \pi$

or other halves: e.g., $14.3 \le \rho \le 15, \ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \ 0 \le \phi \le \pi$

In order to convert triple integrals into spherical coordinates, we have to know how to convert dV.



Notice the extra volume is almost a rectangular prism with height $\rho\Delta\phi$, length $\Delta\rho$, and width $\rho\sin\phi\,\Delta\theta$

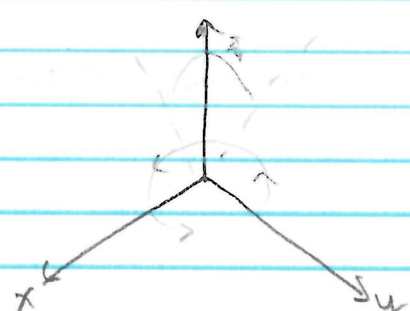
So it has volume $\approx \rho^2 \sin\phi\, \Delta\rho \Delta\theta \Delta\phi$

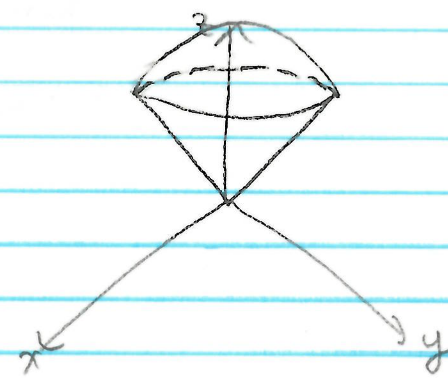
Since this gets more accurate as $\Delta\rho, \Delta\theta, \Delta\phi \to 0$,

$$\boxed{dV = \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi}$$

Ex 5. Sketch the solid whose volume is given by

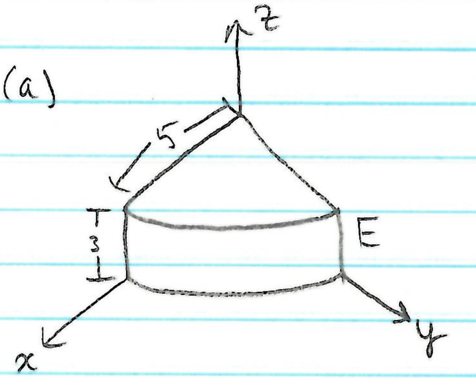
$$\int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$



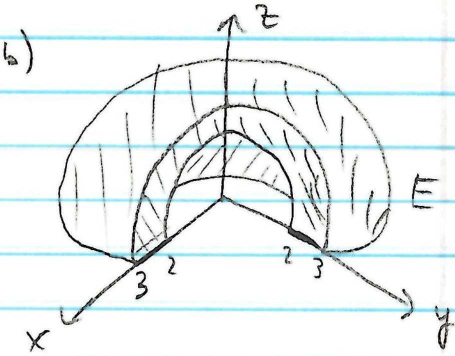


Ex 6. For each of the two solids E below, determine whether it is better to use cylindrical or spherical coordinates and set up $\iiint_E f(x,y,z)\,dV$ in those coordinates.

(a)



(b)



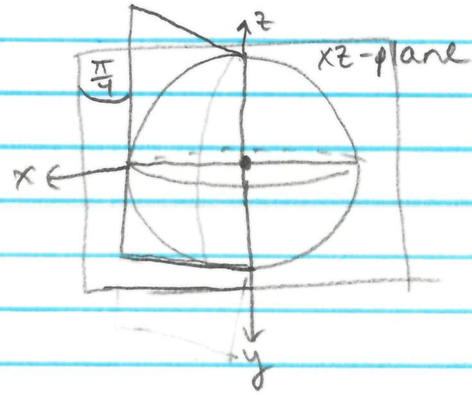
(a) cylindrical (looks like a cylinder)

$$\int_0^3 \int_0^{\frac{\pi}{2}} \int_0^5 f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

(b) spherical (looks like a sphere)

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{2\pi} \int_2^3 f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Ex 7. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\frac{\pi}{4}$.



$$\int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{a} 1\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3\sin\phi\Big|_{\rho=0}^{\rho=a}\,d\theta\,d\phi$$

$$= \frac{a^3}{3}\int_0^{\pi}\int_0^{\pi/4}\sin\phi\,d\theta\,d\phi$$

$$= \frac{a^3\pi}{12}\int_0^{\pi}\sin\phi\,d\phi$$

$$= -\frac{a^3\pi}{12}\cos\phi\Big|_0^{\pi}$$

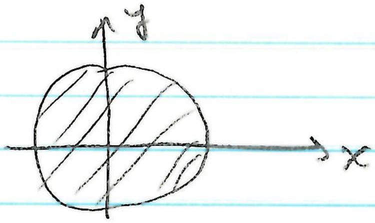
$$= \boxed{\frac{a^3\pi}{12}}$$

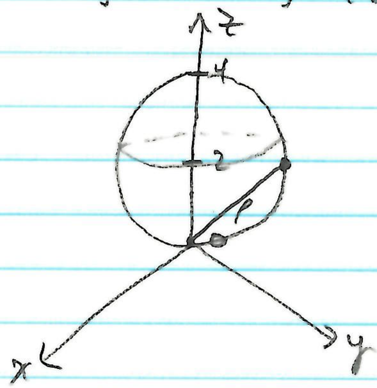
Ex 8. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$

sphere of radius 2 shifted up 2 units

since $z = 2 \pm \sqrt{4-x^2-y^2} \Rightarrow z-2 = \pm\sqrt{4-x^2-y^2}$

$\Rightarrow (z-2)^2 = 4-x^2-y^2 \Rightarrow x^2+y^2+(z-2)^2 = 4$





$x^2+y^2+(z-2)^2=4 \Rightarrow \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi + (\rho\cos\phi-2)^2 = 4$

$\Rightarrow \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\phi - 4\rho\cos\phi + 4 = 4$

$\Rightarrow \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) + \rho^2\cos^2\phi - 4\rho\cos\phi = 0$

$\Rightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi - 4\rho\cos\phi = 0$

$\Rightarrow \rho^2(\sin^2\phi+\cos^2\phi) - 4\rho\cos\phi = 0$

$\Rightarrow \rho^2 - 4\rho\cos\phi = 0$

$\Rightarrow \rho - 4\cos\phi = 0$

$\Rightarrow \rho = 4\cos\phi$

ρ varies from 0 to $4\cos\phi$

θ varies from 0 to 2π

ϕ varies from 0 to $\frac{\pi}{2}$

$(x^2+y^2+z^2)^{3/2} = (\rho^2)^{3/2} = \rho^3$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{4\cos\phi} \rho^3\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{1}{6}\rho^6\sin\phi \Big|_{\rho=0}^{\rho=4\cos\phi} d\theta\, d\phi$$

$$= \frac{2048}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos^6\phi\sin\phi\, d\theta\, d\phi$$

$$= \frac{4096}{3}\pi\int_0^{\frac{\pi}{2}} \cos^6\phi\sin\phi\, d\phi$$

$u = \cos\phi$, $du = -\sin\phi\, d\phi$

$$= \frac{-4096}{3}\pi\int_1^0 u^6\, du$$

$$= \frac{-4096}{3}\pi\cdot\frac{1}{7}u^7\Big|_1^0$$

$$= \boxed{\frac{4096}{21}\pi}$$