MA 261 – Lesson 17

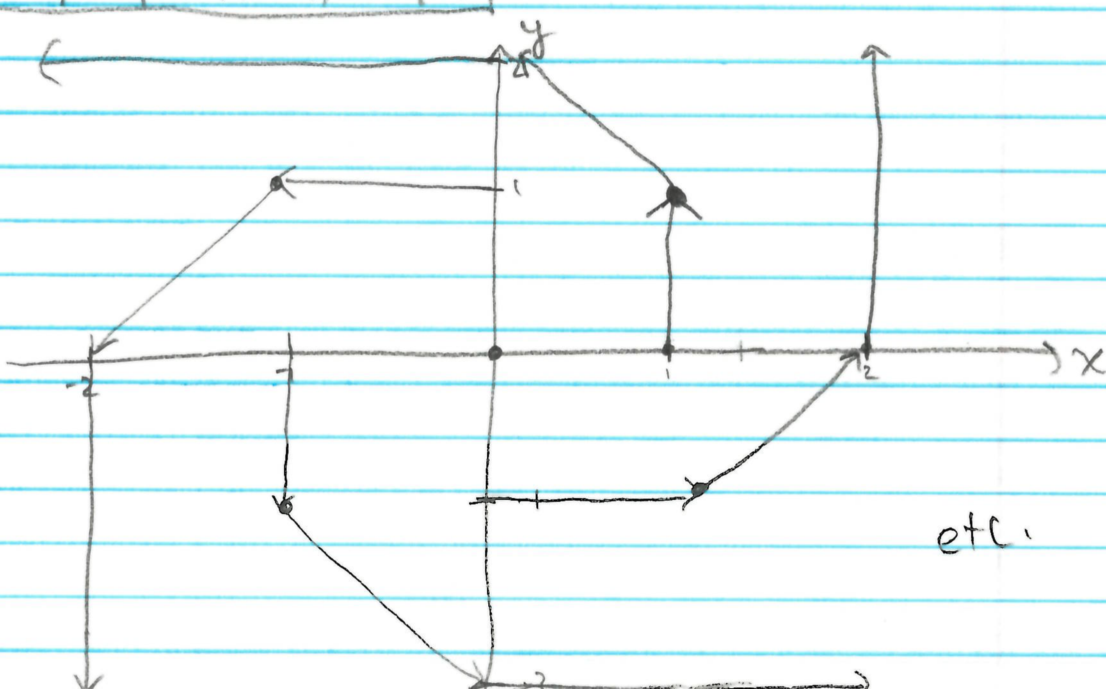
Vector Fields (16.1)

Vector fields are used in physics all the time. A vector field is a function $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$) which assigns to each point in $\mathbb{R}^2$ (or $\mathbb{R}^3$) a vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Usually, we picture vector fields by plotting some of the vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$) with their tails at the point.

Ex 1. Sketch a plot of the vector field $\vec{F}(x,y) = \langle -y, x \rangle$ by sketching some vectors.

x \ y	-2	-1	0	1	2
-2	$\langle 2,-2 \rangle$	$\langle 1,-2 \rangle$	$\langle 0,-2 \rangle$	$\langle -1,-2 \rangle$	$\langle -2,-2 \rangle$
-1	$\langle 2,-1 \rangle$	$\langle 1,-1 \rangle$	$\langle 0,-1 \rangle$	$\langle -1,-1 \rangle$	$\langle -2,-1 \rangle$
0	$\langle 2,0 \rangle$	$\langle 1,0 \rangle$	$\langle 0,0 \rangle$	$\langle -1,0 \rangle$	$\langle -2,0 \rangle$
1	$\langle 2,1 \rangle$	$\langle 1,1 \rangle$	$\langle 0,1 \rangle$	$\langle -1,1 \rangle$	$\langle -2,1 \rangle$
2	$\langle 2,2 \rangle$	$\langle 1,2 \rangle$	$\langle 0,2 \rangle$	$\langle -1,2 \rangle$	$\langle -2,2 \rangle$



etc.

Computers are much better at plotting vector fields since they can plot lots of sample vectors. They often shorten the lengths (to fit on screen) but keep them in the same proportion.

See the plot on the online plotter on my webpage.

Vector fields are useful in physics because they could represent velocity fields or gravitational fields or electric fields, etc.

Throughout this chapter we will deal with things like this.

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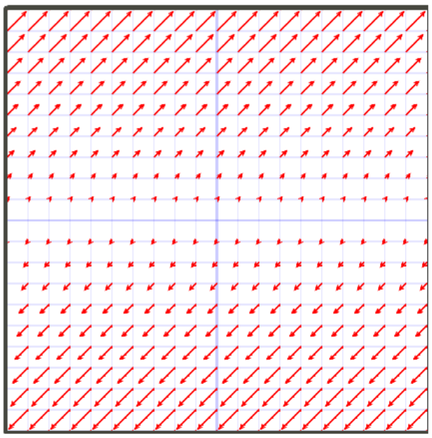
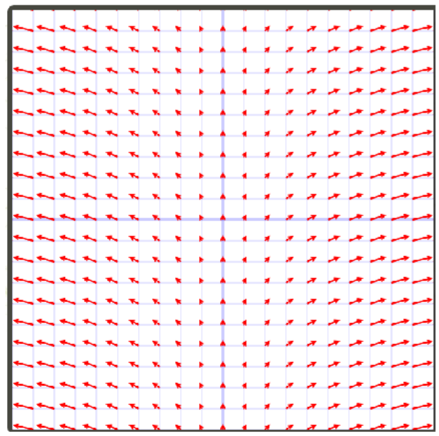
Ex 2. Match the following vector fields (A-D) with their plots (I-IV).

A. $\vec{F}(x,y) = \langle 3, x \rangle$ B. $\vec{F}(x,y) = \langle y, y \rangle$ C. $\vec{F}(x,y) = \langle x, 2 \rangle$ D. $\vec{F}(x,y) = \langle \ln\left(x^2\right), y \rangle$

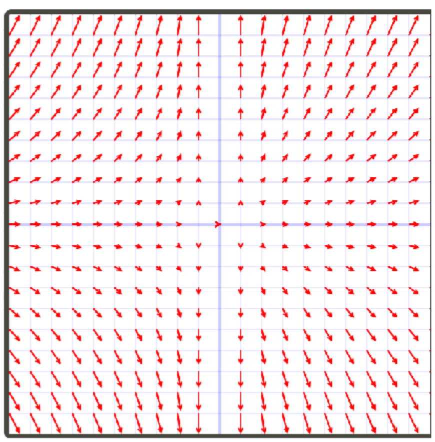
I



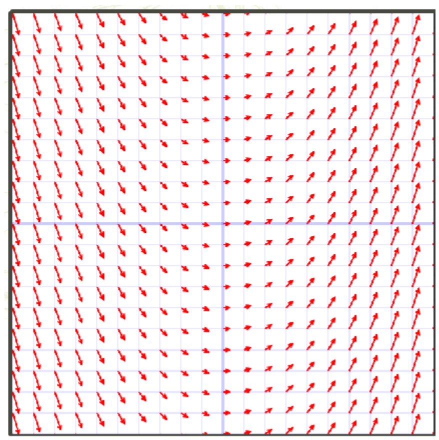
II



III



IV



(Vector fields made in the Kevin Mehall vector field plotter)

(Answer on next page)

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For A, notice that the y-component only depends on the vector's x-position. Hence, all vertical lines should have the same vector along them. Notice also that if x is positive, then the vectors point up and to the right. If x is negative, vectors point down and to the right. This happens with vector field IV.

For B, notice that both the x- and y-components only depend on the vector's y-position, so all horizontal lines should have the same vector along them. Notice also that if y is positive, the vectors should point up and to the right. If y is negative, the vectors should point down and to the left. This happens with vector field I.

For C, notice that the x-component depends only on the x-position, and the y-component is constant. When x is positive, vectors point right and up. When x is negative, vectors point left and up. This happens with vector field II.

For D, notice that the x-component is undefined when $x = 0$ and that the vectors should be identical on opposite sides of the y-axis. This happens with vector field III.

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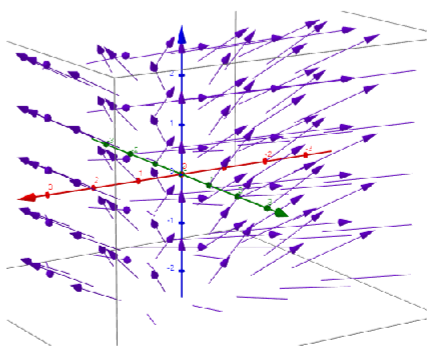
<u>Ex 3</u>. Match the following vector fields (A-D) with their plots (I-IV).

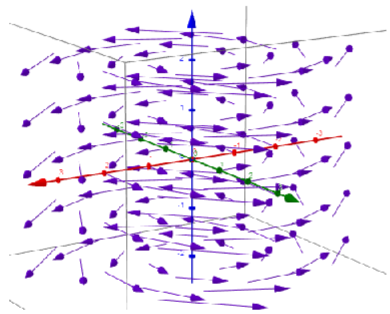
A. $\vec{F}(x, y, z) = \langle -y, x, 0\rangle$ B. $\vec{F}(x, y, z) = \langle z, y, x\rangle$

C. $\vec{F}(x, y, z) = \langle \cos z, 1, -1\rangle$ D. $\vec{F}(x, y, z) = \langle x, y^2, 1\rangle$

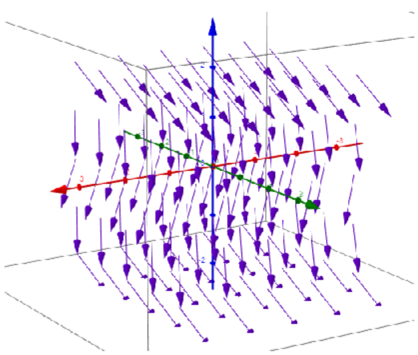
I



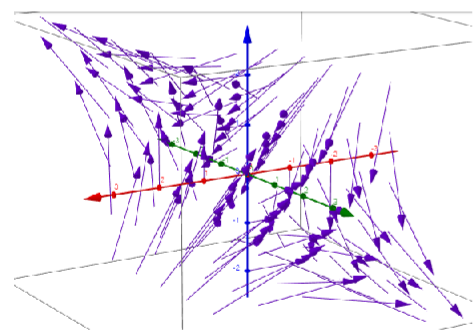
II



III



IV



(Vector fields made in the Geogebra 3d vector field plotter)
(Answer on next page)

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For A, notice that the z-component is 0. Hence, all vectors are horizontal. This happens in vector field II.

For C, notice that the y and z values are both constant, and the x-value depends on the cosine of the z-value. Hence the x-value should oscillate, and lines perpendicular to the yz-plane should all have the same vector. This happens in vector field III.

For D, notice that the y- and z-components are always nonnegative, so all the vectors should be pointing in the positive y-direction and positive x-direction. This happens with vector field I.

By process of elimination, B must be IV. Notice also that the direction the vector is pointing depends on which octant the vector is, and the vectors get larger as you move away from the origin, which is exhibited in IV.

A vector field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ has <u>component functions</u> $P(x,y)$ and $Q(x,y)$. P and Q are sometimes also called <u>scalar fields</u> (since they output a scalar for each point in space).

An important type of vector field is a <u>gradient vector field</u>: where $F(x,y) = \nabla f(x,y)$ for some function $f(x,y)$.

A vector field $\vec{F}$ is called a <u>conservative vector field</u> if it is the gradient vector field for some function f, and f is called a <u>potential function</u> for $\vec{F}$.

In future lessons, we will see how to tell if a vector field is conservative.

<u>Ex 4</u>. Find the gradient vector field of $f(x,y) = x^2y + \sin(y) - \ln|x|$

$$\boxed{\nabla f(x,y) = \left\langle 2xy - \frac{1}{x},\ x^2 + \cos(y) \right\rangle}$$