MA 261 – Lesson 18
Line Integrals (16.2)

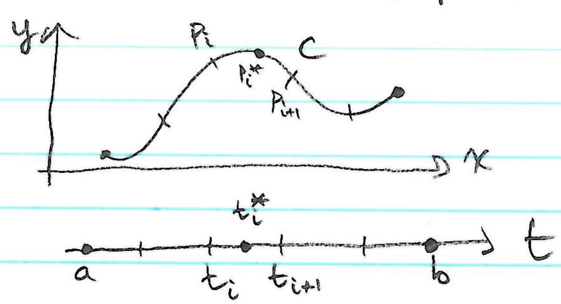
Sometimes we care about the area under a surface above a curve C. (Applications later in lesson)

Finding such a thing is called a "line integral" of a function f along C.

(Sometimes in the past, the word "line" was used to mean a "curve" with a "straight line" being what we usually refer to as a "line" today.)

But how could we possibly compute something like this? We only know how to take an integral over linear curves (like intervals).

But, as we saw earlier, we can turn any smooth plane or space curve into a single variable function over an interval by parameterizing it.



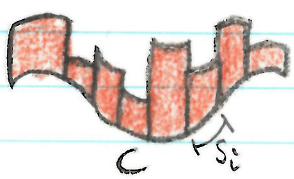
C given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

If s_i represents the length of C along the interval $t_i \le t \le t_{i+1}$, then the area under f above C on this t interval can be approximated by $f(x_i^*, y_i^*)s_i$, where (x_i^*, y_i^*) is the point on C corresponding to the sample point t_i^*.

(We need the length of the curve, just like we did for single integrals)

Then, the area is approximated by the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) s_i$$



(adding areas of curvy "rectangles" above C - notice the importance of having s_i to get the area!)

And the approximation gets arbitrarily good as $s_i \to 0$ (as the t_i get closer, as $n \to \infty$)

Hence, the area is given by

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) s_i$, which we denote

$$\int_C f(x,y)\, ds$$

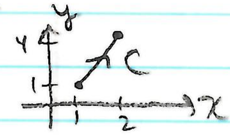
If $x = x(t)$, $y = y(t)$, $a \le t \le b$, is a parameterization of C, then $s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$

so $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

giving $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Hence, $\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Ex 1. Compute $\int_C (x^2+y)\,ds$, where C is the line segment from $(1,1)$ to $(2,4)$.



Remember! The line segment from a point (a,b,c) to a point (c,d,e) can be parameterized by the vector function $\vec{r}(t) = (1-t)\langle a,b,c\rangle + t\langle c,d,e\rangle$, $0 \le t \le 1$ (see Lesson 2)

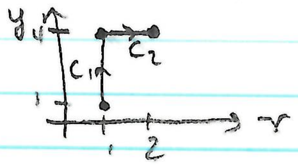
So C: $(1-t)\langle 1,1\rangle + t\langle 2,4\rangle = \langle 1-t, 1-t\rangle + \langle 2t, 4t\rangle$
$= \langle 1+t, 1+3t\rangle$ so $x = 1+t$, $y = 1+3t$, $0 \le t \le 1$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(1)^2+(3)^2}\,dt = \sqrt{10}\,dt$$

$$\begin{aligned}\int_C (x^2+y)\,ds &= \int_0^1 \left((1+t)^2+(1+3t)\right)\sqrt{10}\,dt \\ &= \sqrt{10}\int_0^1 (1+2t+t^2+1+3t)\,dt \\ &= \sqrt{10}\int_0^1 (t^2+5t+2)\,dt \\ &= \sqrt{10}\left(\tfrac{1}{3}t^3+\tfrac{5}{2}t+2t\right)\Big|_0^1 \\ &= \sqrt{10}\left(\tfrac{1}{3}+\tfrac{5}{2}+2\right) \\ &= \boxed{\tfrac{29}{6}\sqrt{10}}\end{aligned}$$

Notice: The curve C matters. What if we took a different path from $(1,1)$ to $(2,4)$?

Ex 2. Compute $\int_C (x^2+y)\,ds$, where C consists of the line segments from (1,1) to (1,4) and from (1,4) to (2,4).



$C_1: x=1,\ y=t,\ 1 \le t \le 4$

$C_2: x=t,\ y=4,\ 1 \le t \le 2$

On C_1, $ds = \sqrt{0+1^2}\,dt = dt$

On C_2, $ds = \sqrt{1^2+0}\,dt = dt$

$$\int_C (x^2+y)\,ds = \int_{C_1} (x^2+y)\,ds + \int_{C_2} (x^2+y)\,ds$$

$$= \int_1^4 \left((1)^2+t\right)dt + \int_1^2 \left((t)^2+4\right)dt$$

$$= \left(t+\tfrac{1}{2}t^2\right)\Big|_1^4 + \left(\tfrac{1}{3}t^3+4t\right)\Big|_1^2$$

$$= \left(4+8-1-\tfrac{1}{2}\right) + \left(\tfrac{8}{3}+8-\tfrac{1}{3}-4\right) = \boxed{\frac{101}{6}}$$

Mass and Center of Mass of a Wire

For similar reasons as we saw in Lesson 13, if you have a wire with density $\rho(x,y,z)$ along a curve C, then

$$\text{mass} = m = \int_C \rho(x,y,z)\,ds$$

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,ds$$

$$\bar{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,ds$$

$$\bar{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,ds$$

where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$
for 3-variables.

Sometimes we don't want to compute a line integral with respect to arc length (ds) but rather with respect to x or y (or z).

These are given by $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$, respectively. These can be done with parameterization (using $dx = \frac{dx}{dt}\,dt$, $dy = \frac{dy}{dt}\,dt$, where $x = x(t)$, $y = y(t)$, $a \le t \le b$ is a parameterization of C) or by replacing one variable with a function of the other, representing C.

Sometimes we may want to do both at once, so we use the notation

$\int_C f(x,y)\,dx + g(x,y)\,dy$ to denote

$$\int_C f(x,y)\,dx + \int_C g(x,y)\,dy$$

Ex 3. Compute $\int_C x\,dy + y^2\,dx$ where C is the part of the parabola $y = x^2$ from $(2,4)$ to $(3,9)$.

If $y = x^2$, then $dy = 2x\,dx$

$$\int_C x\,dy + y^2\,dx = \int_2^3 x \cdot 2x\,dx + \int_2^3 (x^2)^2\,dx$$

$$= \int_2^3 (x^4 + 2x^2)\,dx$$

$$= \frac{1}{5}x^5 + \frac{2}{3}x^3\Big|_2^3$$

$$= \frac{243}{5} + 18 - \frac{32}{5} - \frac{16}{3}$$

$$= \boxed{\frac{823}{15}}$$

Line Integrals of Vector Fields/Work

Suppose you have a particle moving along a curve $C = \vec{r}(t)$ through a force field $\vec{F}$. How much work is done by $\vec{F}$ on the particle?

The work required to move a particle from point P to point Q is $\vec{F} \cdot \vec{PQ}$ (provided that the force is constant)

Breaking the vector function $\vec{r}(t)$ into n subintervals and choosing sample points for each, we approximate the work as

$$W \approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i^*)) \cdot \vec{T}(t_i^*) \Delta s_i$$

where $\vec{T}$ is the unit tangent vector to C.

Hence, $W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{T}(t)\, ds$

But $ds = |\vec{r}\,'(t)|\, dt$ and $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} dt$

So $\boxed{W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$

We sometimes denote this as $\boxed{\int_C \vec{F} \cdot d\vec{r}}$, and this is how we take the line integral of a vector field $\vec{F}$ over the curve $C = \vec{r}(t)$.

Notice: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then
$d\vec{r} = \langle f'(t)\, dt, g'(t)\, dt, h'(t)\, dt \rangle$

Also notice $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$
where $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

Ex 4. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = x\hat{\imath} + 2\hat{\jmath} + zy\hat{k}$ and $\vec{r}(t) = t^2\hat{\imath} + t\hat{\jmath} - 3t\hat{k}$, $0 \le t \le 1$

$$\vec{F}(\vec{r}(t))\cdot d\vec{r} = \langle t^2, 2, -3t^2 \rangle \cdot \langle 2t\,dt, dt, -3\,dt\rangle$$
$$= (2t^3 + 2 + 9t^2)\,dt$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (2t^3 + 2 + 9t^2)\,dt$$

— or —

$$\int_C \vec{F}\cdot d\vec{r} = \int_C x\,dx + 2dy + zy\,dz \quad \text{where } x = t^2, y = t, z = -3t, 0 \le t \le 1$$
$$= \int_0^1 \left(t^2 \cdot 2t\,dt + 2\,dt + (-3t)(t)(-3dt)\right)$$
$$= \int_0^1 (2t^3 + 2 + 9t^2)\,dt$$

$$= \tfrac{1}{2}t^4 + 2t + 3t^3 \Big|_0^1 = \tfrac{1}{2} + 2 + 3 = \boxed{\tfrac{11}{2}}$$

Ex 5. Find the work done by the force field $\vec{F}(x,y) = \langle y, 2x\rangle$ on the particle that moves once around the unit circle counterclockwise

$\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + 2x\,dy = \int_0^{2\pi} \sin t(-\sin t\,dt) + 2\cos t(\cos t\,dt)$$
$$= \int_0^{2\pi} (2\cos^2 t - \sin^2 t)\,dt$$

$$= \int_0^{2\pi} \left(2\cdot\tfrac{1}{2}(1+\cos 2t) - \tfrac{1}{2}(1-\cos 2t)\right) dt$$
$$= \int_0^{2\pi} \left(1 + \cos 2t - \tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right) dt$$
$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2t\right) dt$$
$$= \tfrac{1}{2}t + \tfrac{3}{4}\sin 2t \Big|_0^{2\pi}$$
$$= \pi + 0 - 0 - 0$$
$$= \boxed{\pi}$$