MA 261 - Lesson 20

Green's Theorem (16.4)

We often want to consider \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is a closed curve (starts and ends at the same point).

As we saw in Lesson 19, if \( \vec{F} \) is conservative, then \( \int_C \vec{F} \cdot d\vec{r} = 0 \), but this is generally false if \( \vec{F} \) is not conservative.

Green's Theorem gives a way to compute \( \int_C \vec{F} \cdot d\vec{r} \) when \( C \) is a closed curve. But first, we need to talk about orientation of a closed curve.

A closed curve is **positively oriented** if it is generally traversed counterclockwise, i.e., if the region \( D \) enclosed by the curve is always to the left of a particle traveling along the curve.

A closed curve is **negatively oriented** if the opposite is true.

(Terminology comes from the right hand rule
- if you curl your fingers CCW, thumb points in positive z-direction; if you curl your fingers CW, thumb points in negative z-direction)
Green's Theorem. Let $C$ be a positively oriented, smooth, simple (no self-intersections), closed curve in the $xy$-plane, and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on $D$, then
\[
\int_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA
\]

If $C$ is positively oriented, we often write
\[
\int_C F \cdot d\mathbf{r} \text{ as } \int_C F \cdot d\mathbf{r} \text{ or as } \int_C F \cdot d\mathbf{r}
\]

Note: If $C$ is traversed in a negatively oriented direction, then
\[
\int_C P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy = -\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA
\]

Ex 1. Evaluate $\int_C x^2y^2\,dx + xy\,dy$ where $C$ consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$ in two ways: (a) using the definition, (b) using Green's Theorem.

(a) $C_1$ is parameterized as $\mathbf{r}(x) = (x, x^2), 0 \leq x \leq 1$ $C_2$ is parameterized as $\mathbf{r}(x) = (x, 1), 0 \leq x \leq 1$ in the right-to-left direction $C_3$ is parameterized as $\mathbf{r}(x) = (0, y), 0 \leq y \leq 1$ in the top to bottom direction

On $C_1$, $dx = dx$, $dy = 2x\,dx$
On $C_2$, $dx = dx$, $dy = 0$
On $C_3$, $dx = 0$, $dy = dy$
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\[ \oint_C x^2y^2 \, dx + xy \, dy = \int_0^1 x^2(x^2y^2) \, dx + x(x^2y^2) \cdot 2 \cdot dx \int_0^1 \]
\[ + \int_0^1 x^2(1)^2 \, dx + 0 \int_0^1 \]
\[ + \int_0^1 0 + 0 \cdot y \, dy \int_0^1 \]
\[ = \int_0^1 (x^6 + 2x^4) \, dx - \int_0^1 x^2 \, dx \]
\[ = \frac{3}{4} x^7 + \frac{3}{5} x^5 - \frac{1}{3} x^3 \bigg|_0^1 = \frac{3}{4} + \frac{3}{5} - \frac{1}{3} = \frac{22}{70} \]

(b) \[ \oint_C x^2y^2 \, dx + xy \, dy = \iint_D \left( \frac{\partial}{\partial x} \left[ x^2 \right] - \frac{\partial}{\partial y} \left[ y^2 \right] \right) \, dA \]
\[ = \iint_D (y - 2x^2y) \, dA \]
\[ = \int_0^1 \int_0^1 (y - 2x^2y) \, dy \, dx \]
\[ = \int_0^1 \left( \frac{1}{2} y^2 - x^2 y^2 \right) \bigg|_0^1 \, dx \]
\[ = \int_0^1 \left( \frac{1}{2} - x^2 - \frac{1}{2} x + x^6 \right) \, dx \]
\[ = \frac{1}{4} x - \frac{1}{3} x^3 - \frac{1}{10} x^5 + \frac{1}{7} x^6 \bigg|_0^1 \]
\[ = \frac{1}{4} - \frac{1}{3} - \frac{1}{10} + \frac{1}{7} = \frac{22}{70} \]

Ex 2. Use Green's Theorem to evaluate
\[ \iint_D \vec{F} \cdot d\vec{r} \text{ where } \vec{F}(x,y) = \langle \sqrt{x^2 + y^2}, \tan^{-1} x \rangle \] and C
\[ \text{is the triangle from (0,0) to (1,1) to (1,0) to (0,0).} \]

\[ \text{Notice: C is negatively oriented,} \]
\[ \text{so } \iint_C \vec{F} \cdot d\vec{r} = -\iint_D \left( \frac{\partial}{\partial x} \left[ \tan^{-1} x \right] - \frac{\partial}{\partial y} \left[ \sqrt{x^2 + y^2} \right] \right) \, dA \]
\[ = -\iint_D \left( \frac{1}{1 + x^2} \right) \, dA \]
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \int_0^{\sqrt{x}} \frac{1}{1 + x^2} \, dy \, dx \]

\[ = - \int_0^1 \frac{y}{1 + x^2} \bigg|_{y=0}^{y=x} \, dx \]

\[ = - \int_0^1 \frac{x}{1 + x^2} \, dx \]

\[ u = 1 + x^2, \quad du = 2x \, dx \]

\[ = -\frac{1}{2} \int_1^{1+1} \frac{1}{u} \, du \]

\[ = -\frac{1}{2} \ln|1| - \frac{1}{2} \ln(2) \]

Ex 3. Find \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x,y) = \left(-\frac{1}{2}y, \frac{1}{2}x\right) \)

and \( C \) is the circle \( (x-3)^2 + (y+5)^2 = 1 \)

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_D \nabla \cdot (\nabla \times \mathbf{F}) \, dA \]

\[ = \int_D \left( \frac{1}{2} - \left(-\frac{1}{2}\right) \right) \, dA \]

\[ = \int_D 1 \cdot dA \]

\[ = \text{Area of } D \]

\( D \) is a circle of radius 1, so area is \( \pi \).

Thus, \( \int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\pi} \)
Green’s Theorem proves the result last time about conservative vector fields.

Suppose $\vec{F}(x,y) = P\hat{i} + Q\hat{j}$ with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on an open simply connected region. Let $C$ be any closed curve in the region and $D$ the region bounded by $C$.

By Green’s Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \pm \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \pm \iint_D 0 \, dA = 0$$

Hence, $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve $C$, which is equivalent to $\vec{F}$ being conservative.

Although you won’t see it on your homework, sometimes Green’s Theorem is useful the other way.

For example, suppose you want to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $D$ be the interior. Then

$$A = \iint_D 1 \, dA = \iint_D \left( \frac{\partial}{\partial x} \left[ \frac{1}{2} x \right] - \frac{\partial}{\partial y} \left[ -\frac{1}{2} y \right] \right) dA$$

$$= \int_C -\frac{1}{2} y \, dx + \frac{1}{2} x \, dy = \frac{1}{2} \int_C x \, dy - y \, dx$$

The ellipse can be parameterized as $\vec{r}(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$ so

$$\frac{1}{2} \int_C x \, dy - y \, dx = \frac{1}{2} \int_0^{2\pi} a \cos t \cdot b \sin t \, dt = a b \int_0^{2\pi} \sin t \cos t \, dt = \frac{1}{2} a b \int_0^{2\pi} 2 \sin t \cos t \, dt = \frac{1}{2} a b \cdot 2\pi = a b \pi$$