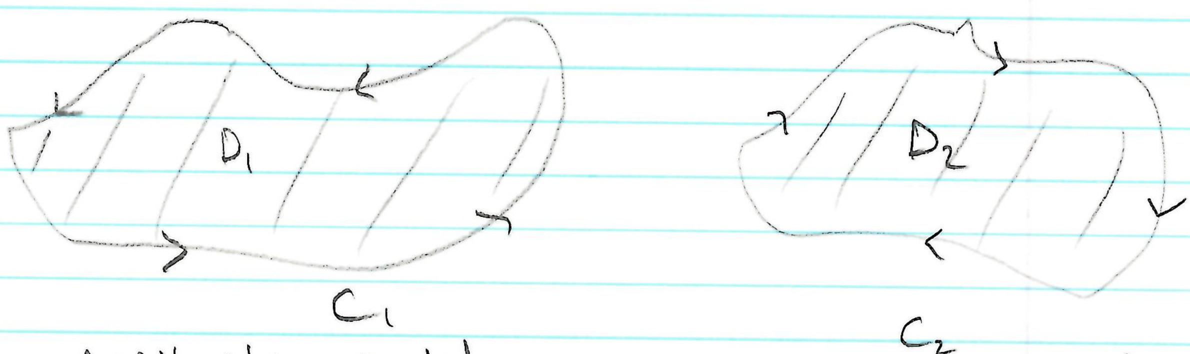
MA 261 – Lesson 20

Green's Theorem (16.4)

We often want to consider $\int_C \vec{F} \cdot d\vec{r}$ where C is a closed curve (starts and ends at the same point).
As we saw in Lesson 19, if $\vec{F}$ is conservative, then $\int_C \vec{F} \cdot d\vec{r} = 0$, but this is generally false if $\vec{F}$ is not conservative.

Green's Theorem gives a way to compute $\int_C \vec{F} \cdot d\vec{r}$ when C is a closed curve. But first, we need to talk about orientation of a closed curve.

A closed curve is <u>positively oriented</u> if it is generally traversed counter-clockwise, i.e., if the region D enclosed by the curve is always to the left of a particle traveling along the curve.
A closed curve is <u>negatively oriented</u> if the opposite is true.



positively oriented

negatively oriented

(Terminology comes from the right hand rule
– if you curl your fingers CCW, thumb points in positive z-direction; if you curl your fingers CW, thumb points in negative z-direction)

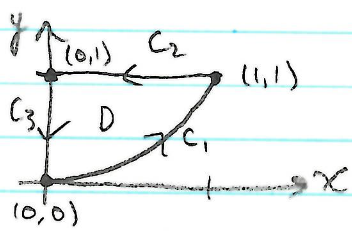
Green's Theorem. Let C be a positively oriented, smooth, simple (no self-intersections), closed curve in the xy-plane, and let D be the region bounded by C. If P and Q have continuous partial derivatives on D, then $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

If C is positively oriented, we often write
$\int_C \vec{F} \cdot d\vec{r}$ as $\oint_C \vec{F} \cdot d\vec{r}$ or as $\oint_C \vec{F} \cdot d\vec{r}$

Note: If C is traversed in a negatively oriented direction, then
$$\int_C P\,dx + Q\,dy = -\oint_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Ex 1. Evaluate $\oint_C x^2y^2\,dx + xy\,dy$ where C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$ in two ways: (a) using the definition, (b) using Green's Theorem.

(a)



C_1 is parameterized as $\vec{r}(x) = \langle x, x^2 \rangle$, $0 \le x \le 1$

C_2 is parameterized as $\vec{r}(x) = \langle x, 1 \rangle$, $0 \le x \le 1$ in the right-to-left direction

C_3 is parameterized as $\vec{r}(x) = \langle 0, y \rangle$, $0 \le y \le 1$ in the top to bottom direction

On C_1, $dx = dx$, $dy = 2x\,dx$
On C_2, $dx = dx$, $dy = 0$
On C_3, $dx = 0$, $dy = dy$

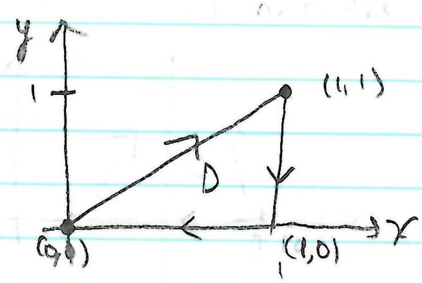
$$\oint_C x^2y^2\,dx + xy\,dy = \int_0^1 x^2(x^2)^2\,dx + x(x^2)\cdot 2x\,dx \quad]\ C_1$$
$$+ \int_1^0 x^2(1)^2\,dx + 0 \quad]\ C_2$$
$$+ \int_1^0 0 + 0\cdot y\,dy \quad]\ C_3$$

$$= \int_0^1 (x^6 + 2x^4)\,dx - \int_0^1 x^2\,dx$$
$$= \tfrac{1}{7}x^7 + \tfrac{2}{5}x^5 - \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{7} + \tfrac{2}{5} - \tfrac{1}{3} = \tfrac{22}{105}$$

(b) $$\oint_C x^2y^2\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^2y^2]\right) dA$$
$$= \iint_D (y - 2x^2y)\,dA$$

$$= \int_0^1\int_{x^2}^1 (y - 2x^2y)\,dy\,dx$$
$$= \int_0^1 \left(\tfrac{1}{2}y^2 - x^2y^2\right)\Big|_{x^2}^1 dx$$
$$= \int_0^1 \left(\tfrac{1}{2} - x^2 - \tfrac{1}{2}x^4 + x^6\right) dx$$
$$= \tfrac{1}{2}x - \tfrac{1}{3}x^3 - \tfrac{1}{10}x^5 + \tfrac{1}{7}x^6\Big|_0^1$$
$$= \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{10} + \tfrac{1}{7} = \tfrac{22}{105}$$

Ex 2. Use Green's Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \langle \sqrt{x^2+1}, \tan^{-1}x\rangle$ and C is the triangle from $(0,0)$ to $(1,1)$ to $(1,0)$ to $(0,0)$.



Notice: C is negatively oriented, so $$\int_C \vec{F}\cdot d\vec{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= -\iint_D \left(\frac{\partial}{\partial x}[\tan^{-1}x] - \frac{\partial}{\partial y}[\sqrt{x^2+1}]\right) dA$$
$$= -\iint_D \left(\frac{1}{1+x^2}\right) dA$$

So $\int_C \vec{F}\cdot d\vec{r} = -\int_0^1 \int_0^x \frac{1}{1+x^2}\,dy\,dx$

$$= -\int_0^1 \left.\frac{y}{1+x^2}\right|_{y=0}^{y=x} dx$$

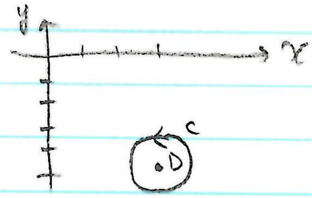
$$= -\int_0^1 \frac{x}{1+x^2}\,dx$$

$u = 1+x^2, \quad du = 2x\,dx$

$$-\frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u|\Big|_1^2 = \boxed{-\frac{1}{2}\ln(2)}$$

Ex 3. Find $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle$ and C is the circle $(x-3)^2 + (y+5)^2 = 1$



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial}{\partial x}\left[\frac{1}{2}x\right] - \frac{\partial}{\partial y}\left[-\frac{1}{2}y\right]\right) dA$$

$$= \iint_D \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) dA$$

$$= \iint_D 1 \cdot dA$$

$$= \text{Area of } D$$

D is a circle of radius 1, so area is π.

Thus, $\oint_C \vec{F}\cdot d\vec{r} = \boxed{\pi}$

Green's Theorem proves the result last time about conservative vector fields.

Suppose $\vec{F}(x,y) = P\hat{i} + Q\hat{j}$ with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on an open simply connected region. Let C be any closed curve in the region and D the region bounded by C.

By Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \pm \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \pm \iint 0 \, dA = 0$$

Hence, $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C, which is equivalent to $\vec{F}$ being conservative.

Although you won't see it on your homework, sometimes Green's Theorem is useful the other way.

For example, suppose you want to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let D be the interior. Then

$$A = \iint_D 1 \, dA = \iint_D \left(\frac{\partial}{\partial x}\left[\tfrac{1}{2}x\right] - \frac{\partial}{\partial y}\left[-\tfrac{1}{2}y\right]\right) dA$$

$$= \oint_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

$$= \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

The ellipse can be parameterized as $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$, $0 \le t \le 2\pi$

so $\tfrac{1}{2}\oint_C x\,dy - y\,dx = \tfrac{1}{2}\int_0^{2\pi} a\cos t \cdot b\cos t\,dt - b\sin t(-a\sin t)\,dt$

$$= \tfrac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\,dt$$

$$= \tfrac{1}{2}\cdot ab \cdot 2\pi = ab\pi$$