MA 261 - Lesson 21

Curl and Divergence (16.5)

As we've seen so far, vector fields are extremely important in vector calculus. The Fundamental Theorem for Line Integrals gives us a way to view the gradient of a scalar field as a "derivative". Here, we introduce the notions of curl and divergence, which will serve as a sort of "derivative" for vector fields.

Define the gradient operator ∇ as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

When we "scalar multiply" ∇ with a scalar field f, we get the gradient of f, $\nabla f = \text{grad} f$.

Given a vector field $\vec{F}$, we can also perform operations on ∇ and $\vec{F}$: $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$

<u>Curl</u>.

Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$.

Then we define $\underline{\text{curl}\,\vec{F}} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right)\hat{\imath} - \left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k}$$

Notice: the curl of a vector field is itself a vector field.

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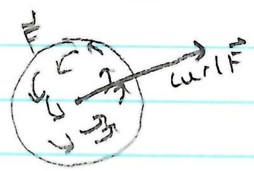
Ex 1. Compute curl $\vec{F}$ where $\vec{F} = x^3yz^2\hat{j} + y^4z^3\hat{k}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}[y^4z^3] - \frac{\partial}{\partial z}[x^3yz^2] \right)\hat{i} - \left(\frac{\partial}{\partial x}[y^4z^3] - \frac{\partial}{\partial z}[0] \right)\hat{j} + \left(\frac{\partial}{\partial x}[x^3yz^2] - \frac{\partial}{\partial y}[0] \right)\hat{k}$$

$$= \boxed{(4y^3z^3 - 2x^3yz)\hat{i} + 3x^2yz^2\,\hat{k}}$$

Intuitively speaking, given a vector field, the curl of the vector field at that point gives a vector representing how much the vector field "curls" around that point.



The magnitude represents the "speed" of the curling, and the direction represents the orientation of the curling.

If $\vec{F}$ is a conservative vector field, then $\text{curl}\,\vec{F} = \vec{0}$.

Proof. $\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)\hat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\hat{k}$$

$= \vec{0}$ by Clairaut's Theorem. ▨

The Converse is also true if $\vec{F}$ is defined everywhere.

Theorem. If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and if $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Ex 2. Determine whether $\vec{F}$ is conservative. If it is, find a potential function.

(a) $\vec{F}(x,y,z) = z\cos y\,\hat{\imath} + xz\sin y\,\hat{\jmath} + x\cos y\,\hat{k}$

(b) $\vec{F}(x,y,z) = \hat{\imath} + \sin z\,\hat{\jmath} + y\cos z\,\hat{k}$

(a) $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos y & xz\sin y & x\cos y \end{vmatrix}$

$= (-x\sin y - x\sin y)\hat{\imath} + \ldots$ stuff

$\hat{\imath}$-component is nonzero, so $\text{curl}\,\vec{F} \neq \vec{0}$.
$\vec{F}$ is not conservative.

(b) $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix}$

$= (\cos z - \cos z)\hat{\imath} - (0-0)\hat{\jmath} + (0-0)\hat{k} = \vec{0}$

so $\vec{F}$ is conservative, since $\vec{F}$ is defined on all of $\mathbb{R}^3$.

$f = \int P\,dx = \int 1\,dx = x + g(y,z)$

$f_y = g_y$ and also $= Q = \sin z$ so $g_y = \sin z$

$g = \int g_y\,dy = \int \sin z\,dy = y\sin z + h(z)$

(can't be a function of x since g has no x-dependence)

$f = x + y\sin z + h(z)$

$f_z = y\cos z + h'(z)$ and also $= R = y\cos z$

$\Rightarrow h'(z) = 0$ so $h(z) = K$

$$\boxed{f(x,y,z) = x + y\sin z + K \quad \text{(can choose } K=0\text{)}.}$$

Divergence

Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$.

We define $\text{div}\,\vec{F} = \nabla \cdot \vec{F}$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

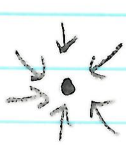
Notice: The divergence of a vector field is a scalar field.

Ex 3. Compute $\text{div}\,\vec{F}$ where $\vec{F}(x,y,z) = x^3yz^2\hat{\jmath} + y^4z^3\hat{k}$.

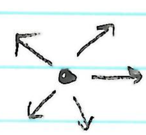
$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}[0] + \frac{\partial}{\partial y}[x^3yz^2] + \frac{\partial}{\partial z}[y^4z^3]$$

$$= \boxed{x^3z^2 + 3y^4z^2}$$

Intuitively, the divergence of a vector field at a point tells you how much the vector field "diverges" from that point. A positive divergence says the point repels the vector field nearby. A negative divergence says the point attracts the vector field nearby.



negative divergence

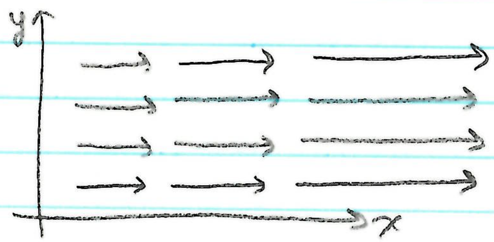


positive divergence

Ex 4. The vector field $\vec{F}$ shown in the xy-plane looks the same in all other horizontal planes. i.e., $\vec{F}$ is independent of z and its z-component is 0.

(a) Is div $\vec{F}$ positive, negative, or zero?

(b) Is curl $\vec{F} = \vec{0}$? If not, what direction does it point?



Notice: The y-component is also 0, so $\vec{F} = P\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

(a) $\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[P] + \frac{\partial}{\partial y}[0] + \frac{\partial}{\partial z}[0] = \frac{\partial P}{\partial x}$

Notice: as x increases, the length of the vectors increase, so $\frac{\partial P}{\partial x} > 0$. Hence div $\vec{F}$ is $\boxed{\text{positive}}$

(b) $\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & 0 & 0 \end{vmatrix}$

$= (0-0)\hat{\imath} - \left(0 - \frac{\partial}{\partial z}P\right)\hat{\jmath} + \left(0 - \frac{\partial}{\partial y}P\right)\hat{k}$

$= -\frac{\partial P}{\partial z}\hat{\jmath} - \frac{\partial P}{\partial y}\hat{k}$

Notice that P depends only on the x-value (changing y or z and fixing x does not change the vector)

so $\frac{\partial P}{\partial z} = 0 = \frac{\partial P}{\partial y}$.

Hence, $\boxed{\operatorname{curl}\vec{F} = \vec{0}}$

Recall:

grad - takes scalar fields, produces vector fields
curl - takes vector fields, produces vector fields
div - takes vector fields, produces scalar fields

Ex 5. Let $\vec{F}$ be a vector field and f be a scalar field. Determine whether each of the following is meaningful. If it is, is the result a vector or scalar field?

(a) grad(curl $\vec{F}$)
(b) grad(div $\vec{F}$)
(c) curl(curl $\vec{F}$)
(d) (grad f) • (curl $\vec{F}$)

(a) meaningless since curl $\vec{F}$ is a vector field and grad only takes scalar fields
(b) meaningful (div $\vec{F}$ is a scalar field, and grad takes scalar fields). The result is a vector field.
(c) meaningful (curl $\vec{F}$ is a vector field, and curl takes vector fields). The result is a vector field.
(d) meaningful (grad f and curl $\vec{F}$ are vector fields, which can be dotted). The result is a scalar field.

Ex 6. Prove $\text{div}(f\vec{F}) = f \,\text{div}\,\vec{F} + \vec{F} \cdot \nabla f$, assuming f is a scalar field and $\vec{F}$ is a vector field and all partial derivatives are continuous.

Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$

Then $f\vec{F} = fP\hat{\imath} + fQ\hat{\jmath} + fR\hat{k}$

$$\begin{aligned} \text{div}(f\vec{F}) &= \frac{\partial}{\partial x}[fP] + \frac{\partial}{\partial y}[fQ] + \frac{\partial}{\partial z}[fR] \\ &= fP_x + f_x P + fQ_y + f_y Q + fR_z + f_z R \end{aligned}$$

Now, $f \,\text{div}\,\vec{F} = f(P_x + Q_y + R_z) = fP_x + fQ_y + fR_z$

$\vec{F} \cdot \nabla f = f_x P + f_y Q + f_z R$

So
$$\begin{aligned} f\,\text{div}\,\vec{F} + \vec{F} \cdot \nabla f &= fP_x + fQ_y + fR_z + f_x P + f_y Q + f_z R \\ &= fP_x + f_x P + fQ_y + f_y Q + fR_z + f_z R \\ &= \text{div}(f\vec{F}). \end{aligned}$$

You will prove a similar "product rule" for curl on your homework.