MA 261 - Lesson 22

Parametric Surfaces (16.6)

Earlier in the course, we saw that curves could be given parametrically with a single parameter:
$x=f(t),\ y=g(t),\ z=h(t)$

Using more parameters, we can actually describe surfaces.

Given a parametric/vector representation of a surface, we can often identify the surface by eliminating the parameter.

Ex 1. Identify the surface given by
$\vec{r}(u,v) = \langle u+3v,\ 1-u+v,\ 3+v\rangle$

Here, we have $x=u+3v,\ y=1-u+v,\ z=3+v$

Hence, we get $v=z-3$

Then $x=u+3(z-3)=u+3z-9$

So $u=x-3z+9$

Then
$$y=1-(x-3z+9)+(z-3)$$
$$y=1-x+3z-9+z-3$$
$$y=-x+4z-11$$
or
$$\boxed{\begin{array}{l}x+y-4z=-11\\ \qquad\text{A plane}\end{array}}$$

Ex 2. Identify the surface given by
$\vec{r}(s,t) = s\cos t\,\hat{\imath} + s^2\,\hat{\jmath} + s\sin t\,\hat{k}$

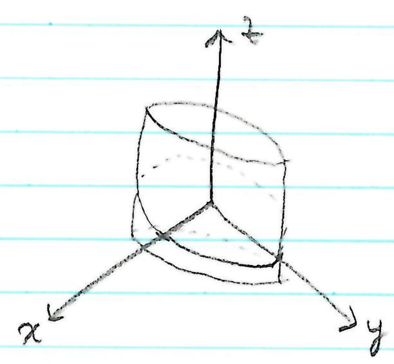
Here, $x = s\cos t$, $y = s^2$, $z = s\sin t$
Hence, $x^2 + z^2 = s^2\cos^2 t + s^2\sin^2 t = s^2 = y$

$y = x^2 + z^2$ is a hyperboloid opening up along the positive y-axis

Ex 3. Identify the surface given by
$\vec{r}(s,t) = \langle 2\cos t, 3\sin t, s\rangle$, $-1 \le s \le 2$

$x = 2\cos t$, $y = 3\sin t$, $z = s$
$(\frac{1}{2}x)^2 + (\frac{1}{3}y)^2 = \cos^2 t + \sin^2 t = 1$
so $\frac{x^2}{4} + \frac{y^2}{9} = 1$

We get part of an elliptic cylinder with curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z=0$ and z between -1 and 2



If you graph these surfaces on a parametric surfaces plotter, you will notice that the surfaces have a kind of "grid" on them.

These are called "grid curves." Given a parametric surface depending on u and v, you get grid curves as the curves obtained by fixing u or by fixing v.

<u>Ex 4</u> Identify the grid curves of
$\vec{r}(u,v) = \langle v, u\cos v, u\sin v\rangle$
and match it with the surface (on the next page) that it gives.

If you fix u, then you have the curve
$\langle v, a\cos v, a\sin v\rangle$ for some constant a.
This has $y^2+z^2=a^2$, a circle in the yz-plane of radius a, but the x-value changes, making it a spiral whose axis is the x-axis

If you fix v, then $\cos v = b$, $\sin v = c$, $v=a$ for some constants a, b, c.
$x=a$, $y=bu$, $z=cu$.
$x=a$, $\frac{y}{b}=\frac{z}{c}$ are symmetric equations of a line passing through the x-axis

The only surface on the next page with grid lines like this is $\boxed{II}$

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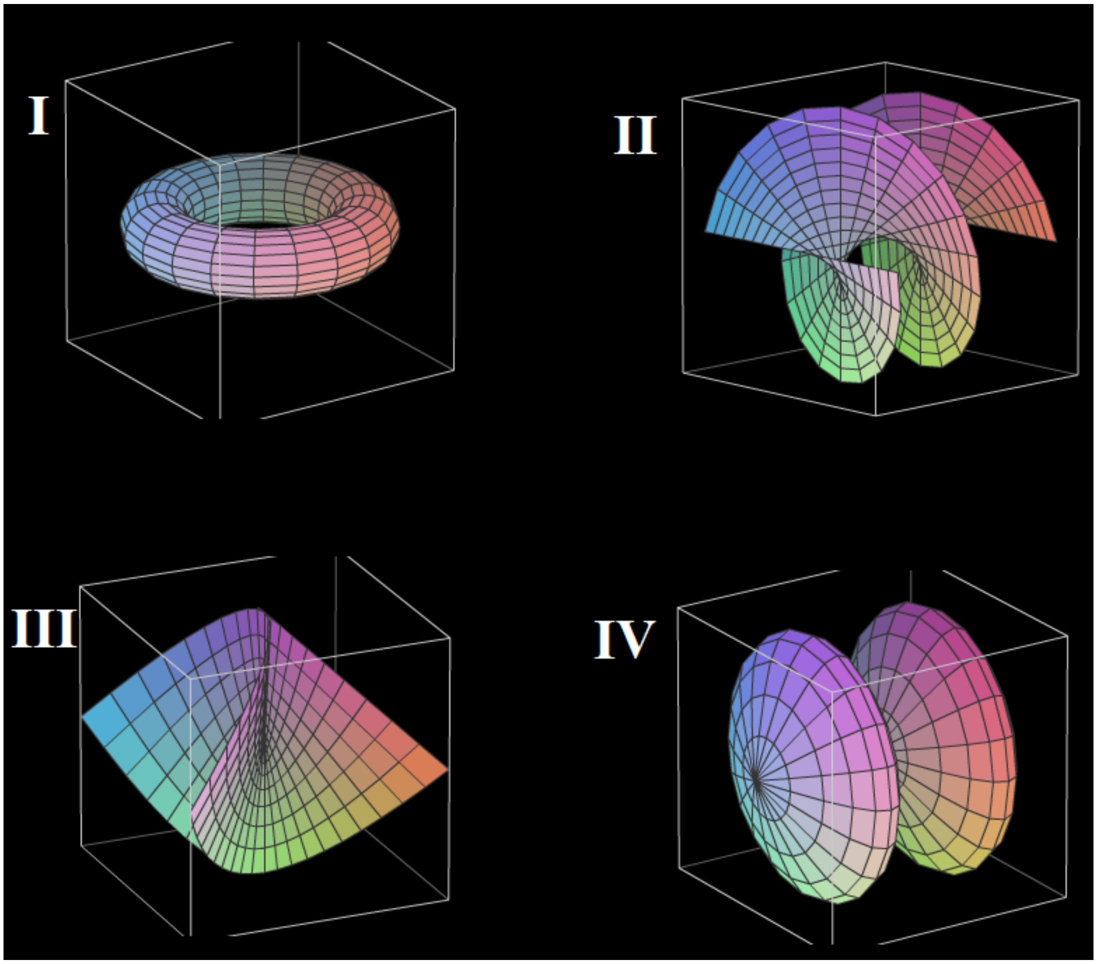
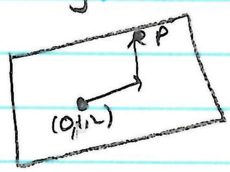


Figure 1: Made in Barbara Kaskosz's parametric surfaces plotter

Ex 5. Find a parametric representation for the plane containing the point $(0,1,2)$ and the vectors $\langle -1, 0, 1\rangle$ and $\langle 2, 1, 0\rangle$.

Notice: any point P on the plane can be achieved by adding scalings of the vectors to the "point."



(Different scalings give different points)

So $\vec{r}(u,v) = \langle 0,1,2\rangle + u\langle -1,0,1\rangle + v\langle 2,1,0\rangle$
$= \langle 2v-u,\ 1+v,\ 2+u\rangle$

$\boxed{x = 2v-u,\quad y = 1+v,\quad z = 2+u}$

Ex 6. Find a parametric representation for the part of the cone $y^2 - x^2 - z^2 = 0$ in front of the xz-plane.

Notice: In front of the xz-plane is where $y \geq 0$.

$y^2 = x^2 + z^2$

$y = \pm\sqrt{x^2+z^2}$

only need positive root, so

$\boxed{x = x,\quad y = \sqrt{x^2+z^2},\quad z = z}$

or

$\boxed{x = u,\quad y = \sqrt{u^2+v^2},\quad z = v}$

Sometimes spherical and cylindrical coordinates can help us get parametric representations of surfaces.

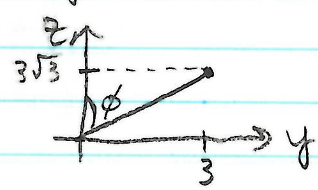
Ex 7. Find a parametric representation for the part of the sphere $x^2+y^2+z^2=36$ that lies between the planes $z=0$ and $z=3\sqrt{3}$.

The sphere generally has equation $\rho=6$ in spherical coordinates.

The plane $z=3\sqrt{3}$ intersects the sphere where

$$x^2+y^2+(3\sqrt{3})^2=36 \Rightarrow x^2+y^2+27=36 \Rightarrow x^2+y^2=9$$

Projecting onto the $x=0$ plane, get $y^2=9 \Rightarrow y=\pm 3$



$\tan\phi = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{(\frac{1}{2})}{(\frac{\sqrt{3}}{2})}$

so $\phi = \frac{\pi}{6}$

The plane $z=0$ intersects the sphere when $\phi=\frac{\pi}{2}$

θ can vary between 0 and 2π

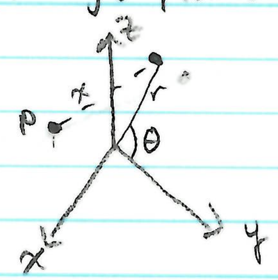
$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$,

where $\rho=6$, $0\le\theta\le 2\pi$, $\frac{\pi}{6}\le\phi\le\frac{\pi}{2}$

$$\boxed{x=6\sin\phi\cos\theta,\ y=6\sin\phi\sin\theta,\ z=6\cos\phi,\ 0\le\theta\le 2\pi,\ \frac{\pi}{6}\le\phi\le\frac{\pi}{2}}$$

Ex 8. Find a parametric representation for the part of the cylinder $y^2 + z^2 = 16$ that lies between the planes $x = 2$ and $x = 5$.

We can set up an alternative version of cylindrical coordinates here: (x, r, θ) where r and θ deal with the yz-plane.



Here, $r^2 = y^2 + z^2 = 16$ so $r = 4$

$y = r\cos\theta$, $z = r\sin\theta$

so $\boxed{x = x,\ y = 4\cos\theta,\ z = 4\sin\theta,\ 2 \le x \le 5,\ 0 \le \theta \le 2\pi}$

or

$\boxed{x = s,\ y = 4\cos\theta,\ z = 4\sin\theta,\ 2 \le s \le 5,\ 0 \le \theta \le 2\pi}$