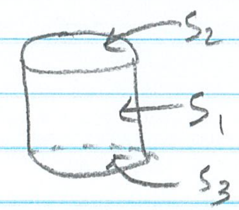
MA 261 - Lesson 24

More on Surface Integrals (16.7)

Just like with line integrals, if a surface can be decomposed into multiple surfaces, the surface integral is the sum of the surface integrals.

Ex 1. Evaluate $\iint_S (x^2+y^2+z^2)\,dS$, where S is the part of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=3$.



S_1: $\langle 2\cos\theta, 2\sin\theta, z\rangle$, $0 \le \theta \le 2\pi$, $0 \le z \le 3$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos\theta\,\hat{i} + 2\sin\theta\,\hat{j}$$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2$$

$$x^2+y^2+z^2 = 2^2\cos^2\theta + 2^2\sin^2\theta + z^2 = 4+z^2$$

$$\iint_{S_1} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (4+z^2)\cdot 2\,dz\,d\theta$$

$$= 2\int_0^{2\pi} \left(4z+\tfrac{1}{3}z^3\right)\Big|_0^3\,d\theta$$

$$= 2\cdot 21\cdot 2\pi = 84\pi$$

S_2: $\langle r\cos\theta, r\sin\theta, 3\rangle$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (r\cos^2\theta + r\sin^2\theta)\,\hat{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = r$$

$$x^2+y^2+z^2 = r^2\cos^2\theta + r^2\sin^2\theta + 3^2 = r^2+9$$

$$\iint_{S_2} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 (r^2+9)r\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{1}{4}r^4+\tfrac{9}{2}r^2\right)\Big|_0^2 \,d\theta = 22\cdot 2\pi = 44\pi$$

S_3 : $\langle r\cos\theta, r\sin\theta, 0\rangle$, $0\le r\le 2$, $0\le\theta\le 2\pi$

$|\vec{r}_\theta \times \vec{r}_r| = r$

$x^2+y^2+z^2 = r^2+0 = r^2$

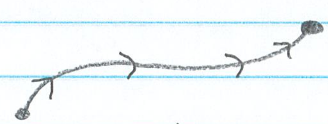
$$\iint_{S_3}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$
$$= \tfrac{1}{4}r^4\Big|_0^2\cdot 2\pi = 8\pi$$

So $\iint_S (x^2+y^2+z^2)\,dS = 84\pi + 44\pi + 8\pi = \boxed{136\pi}$

Oriented Surfaces

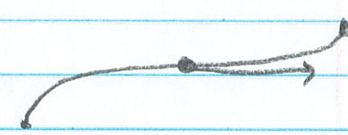
Much like smooth curves can be oriented by direction traversed, some surfaces can also be oriented.

Really, for a smooth curve, to orient it, we are looking at directions of tangent vectors.

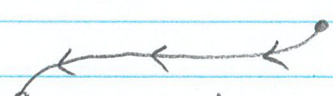


rightward orientation

$\longleftrightarrow$

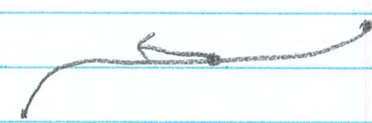


choose tangent vector pointing "right"



leftward orientation

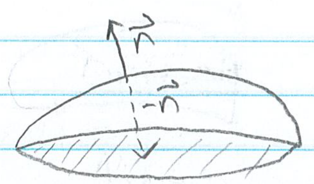
$\longleftrightarrow$



choose tangent vector pointing "left"

To get an orientation, we want the tangent vector to change continuously as we traverse the curve (this guarantees that all tangent vectors point the "same direction" along the curve, since for a smooth curve, $\vec{r}\,'(t) \neq \vec{0}$ for any t)

For a surface, instead of having tangent vectors, we have tangent planes. The "direction" of a plane is given by its normal vectors. Just like a curve, planes have two opposite directions for normal vectors (usually "upward" and "downword")



To get an orientation, we want all normal vectors to point the "same way" and we guarantee this by requiring them to change continuously as the surface is traversed.

"Upward" is where the $\hat{k}$-component is positive
"downward" is where the $\hat{k}$-component is negative.

Given a parametric surface $\vec{r}(u,v)$, a normal vector is given by $\vec{r}_u \times \vec{r}_v$.

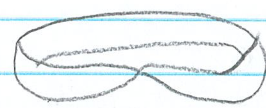
Ex 2. Find a normal vector in the upward orientation for the paraboloid $z = 9 - x^2 - y^2$.

Take $\vec{r}(x,y) = \langle x, y, 9-x^2-y^2 \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{k}$$

$\hat{k}$ - component is positive, so $\boxed{\langle 2x, 2y, 1 \rangle}$

Not all surfaces are orientable. For example, the Möbius Strip.



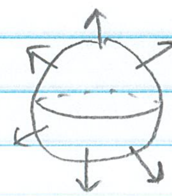
It is a "one-sided" surface

It is impossible to define a normal vector at each point which changes continuously as you traverse the surface.

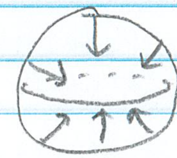
A closed surface is a surface which encloses a finite region of space (e.g., a sphere or ellipsoid). This is like how a closed curve encloses a region of 2-space.

For a closed surface, it doesn't make sense to talk about upward/downward/etc. But you can have inward and outward.

Positive orientation is where all normal vectors point outward; negative orientation is where all the vectors point inward.



positive orientation



negative orientation

Ex 3. Find a normal vector for the sphere of radius 3 centered at the origin with positive orientation.

$$\vec{r}(\theta, \phi) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \end{vmatrix}$$

$$= (-9\sin^2\phi\cos\theta)\hat{i} - (9\sin^2\phi\sin\theta)\hat{j} + (-9\sin\phi\cos\phi)\hat{k}$$

To have positive orientation, the $\hat{k}$-component for the top half $(0 \le \phi \le \frac{\pi}{2})$ must be positive.

When $0 \le \phi \le \frac{\pi}{2}$, $\sin\phi\cos\phi \ge 0$

so $-9\sin\phi\cos\phi \le 0$, this has negative orientation.

so $\boxed{\langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi \rangle.}$

Surface Integrals of Vector Fields / Flux

If $\vec{F}$ is the electric field and S is a surface in $\vec{F}$, then the flux of $\vec{F}$ across S is given by

$$\iint_S \vec{F} \cdot \vec{n} \, dS,$$ where $\vec{n}$ is the unit normal vector of S for some orientation,

which is often denoted $\iint_S \vec{F} \cdot d\vec{S}$

If $\vec{r}(u,v)$ is a parameterization of S, and if $\vec{r}_u \times \vec{r}_v$ is pointing in the correct direction, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

So
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS$$
$$= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, |\vec{r}_u \times \vec{r}_v|\, dA$$

Hence, if $\vec{F}$ is a vector field, S is an oriented surface with parameterization $\vec{r}(u,v)$, D is the region in the uv-plane defining S, and if $\vec{r}_u \times \vec{r}_v$ is in the correct orientation for S, then the surface integral of $\vec{F}$ over S (or the flux integral of $\vec{F}$ across S) is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

Ex 4. Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ with upward orientation.

From Ex 2, $\vec{r}(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$, $0 \le x \le 1$, $0 \le y \le 1$ and the normal vector with upward orientation is

$$\langle 2x, 2y, 1 \rangle.$$

$$\vec{F}(\vec{r}(x,y)) = \langle xy, y(9-x^2-y^2), (9-x^2-y^2)x \rangle = \langle xy, 9y - yx^2 - y^3, 9x - x^3 - xy^2 \rangle$$

$$\vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y) = 2x^2y + 18y^2 - 2y^2x^2 - 2y^4 + 9x - x^3 - xy^2$$

So $\iint_S \vec{F} \cdot d\vec{S}$

$$= \int_0^1 \int_0^1 (2x^2y + 18y^2 - 2x^2y^2 - 2y^4 + 9x - x^3 - xy^2)\, dx\, dy$$

$$= \int_0^1 \left(\tfrac{2}{3}x^3y + 18xy^2 - \tfrac{2}{3}x^3y^2 - 2xy^4 + \tfrac{9}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2y \right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\tfrac{2}{3}y + 18y^2 - \tfrac{2}{3}y^2 - 2y^4 + \tfrac{9}{2} - \tfrac{1}{4} - \tfrac{1}{2}y \right) dy$$

$$= \int_0^1 \left(-2y^4 + \tfrac{52}{3}y^2 + \tfrac{1}{6}y + \tfrac{17}{4} \right) dy$$

$$= -\tfrac{2}{5}y^5 + \tfrac{52}{9}y^3 + \tfrac{1}{12}y^2 + \tfrac{17}{4}y \Big|_0^1$$

$$= -\tfrac{2}{5} + \tfrac{52}{9} + \tfrac{1}{12} + \tfrac{17}{4}$$

$$= \boxed{\tfrac{437}{45}}$$

MA 261 - Lesson 24

Ex 5. Find $\iint_S \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z^2\hat{k}$ where S is the sphere of radius 3 centered at the origin with positive orientation.

From Ex 3, $\vec{r}(\theta, \phi) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$
with $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

and normal vector with positive orientation is
$\vec{n} = \langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi \rangle$.

$\vec{F}(\vec{r}(\theta,\phi)) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, (3\cos\phi)^2 \rangle$

$\vec{F}(\vec{r}(\theta,\phi)) \cdot \vec{n}$

$= 27\sin^3\phi\cos^2\theta + 27\sin^3\phi\sin^2\theta + 81\sin\phi\cos^3\phi$

$= 27(\sin^3\phi + 3\sin\phi\cos^3\phi)$

$\iint_S \vec{F} \cdot d\vec{S}$

$= 27\int_0^{\pi}\int_0^{2\pi} (\sin^3\phi + 3\sin\phi\cos^3\phi)\, d\theta\, d\phi$

$= 27 \cdot 2\pi \int_0^{\pi} (\sin\phi\sin^2\phi + 3\sin\phi\cos^3\phi)\, d\phi$

$= 54\pi \int_0^{\pi} \sin\phi(1 - \cos^2\phi + 3\cos^3\phi)\, d\phi$

$u = \cos\phi$, $du = -\sin\phi\, d\phi$

$= -54\pi \int_1^{-1} (1 - u^2 + 3u^3)\, du$

$= -54\pi \left(u - \frac{1}{3}u^3 + \frac{3}{4}u^4\right)\Big|_1^{-1}$

$= -54\pi\left(-1 + \frac{1}{3} + \frac{3}{4} - 1 + \frac{1}{3} - \frac{3}{4}\right)$

$= -54\pi\left(-\frac{4}{3}\right)$

$= \boxed{72\pi}$