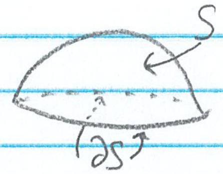
# MA 261 – Lesson 25

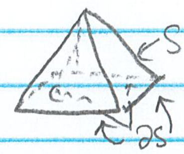
## Stokes' Theorem (16.8)

### Induced Orientation of a Boundary Curve

Suppose you have an oriented surface $S$ with a boundary curve (which we often denote $\partial S$)

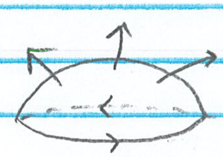


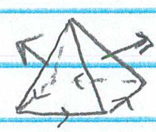

hemisphere without bottom

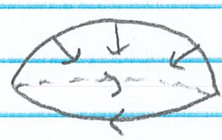


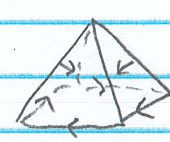

pyramid without bottom.

We can induce an orientation on $\partial S$ based on the orientation on $S$. – use the right hand rule; that is, give $\partial S$ an orientation so that when you curl your fingers in that direction, your thumb points in the general direction of the orientation of $S$.









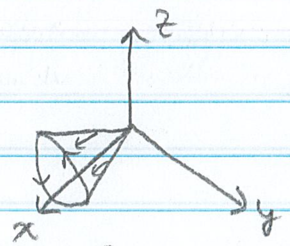
If you view a person as walking around the boundary curve with their feet on the curve and their head pointing in the general direction of the orientation of $S$, the induced orientation on $\partial S$ has them walking around so $S$ is always on their left.

<u>Stokes' Theorem</u>. Let $S$ be an oriented surface with smooth boundary curve $\partial S$ given the induced orientation. Then

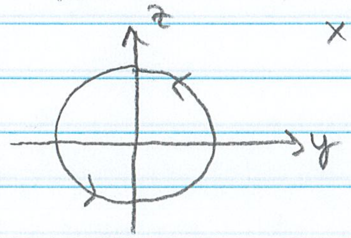
$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

This is like the Fundamental Theorem of Calculus since the integral of a derivative (curl) of $\vec{F}$ over a surface is the same as looking at $\vec{F}$ over the boundary.

<u>Ex 1</u>. Use Stokes' Theorem to evaluate $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$, $\vec{F}(x,y,z) = \tan^{-1}(x^2yz^2)\hat{\imath} + x^2y\hat{\jmath} + x^2z^2\hat{k}$, where $S$ is the cone $x = \sqrt{y^2+z^2}$, $0 \le x \le 2$, oriented in the direction of the positive x-axis.




The boundary curve is $2 = \sqrt{y^2+z^2}$ $\iff y^2+z^2=4,\ x=2$ with ccw orientation viewed from the positive x-direction




x=2 - plane

$\partial S$: $\vec{r}(t) = \langle 2, 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$
$d\vec{r} = \langle 0, -2\sin t, 2\cos t\rangle\, dt$

$$\begin{aligned}\vec{F}(\vec{r}(t)) \cdot d\vec{r} &= 0 + x^2y(-2\sin t\, dt) + x^2z^2(2\cos t\, dt) \\ &= 4 \cdot 2\cos t \cdot (-2\sin t)\, dt + 4 \cdot 4\sin^2 t \cdot 2\cos t\, dt \\ &= (-16\sin t\cos t + 32\sin^2 t\cos t)\, dt\end{aligned}$$

By Stokes' Thm,

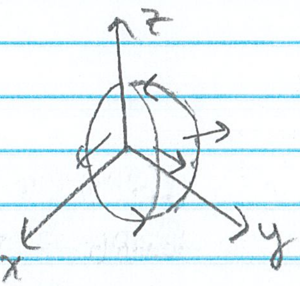
$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\sin t\cos t + 32\sin^2 t\cos t)\, dt$$

Let $u = \sin t$, $du = \cos t\, dt$

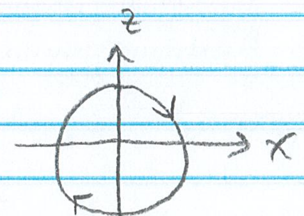
$$\int_0^0 (-16u + 32u^2)\, du = 0$$

So $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \boxed{0}$

Ex 2. Use Stokes' Thm to evaluate $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$, $\vec{F}(x,y,z) = ze^y \hat{\imath} + x\cos y\, \hat{\jmath} + xz \sin y\, \hat{k}$, $S$ is the hemisphere $x^2+y^2+z^2 = 25$, $y \geq 0$, oriented in the direction of the positive $y$-axis.




$\partial S$ is given by $x^2+z^2=25$, $y=0$ which is oriented in the ccw direction viewing it from the positive $y$-axis but the cw direction viewing it in the $xz$-plane!




$$\partial S: \langle 5\cos(-t), 0, 5\sin(-t)\rangle, \quad 0 \leq t \leq 2\pi$$
$$= \langle 5\cos t, 0, -5\sin t\rangle$$

$\vec{F}(\vec{r}(t)) = \langle -5\sin t, 5\cos t, 0\rangle$

$d\vec{r}(t) = \langle -5\sin t, 0, -5\cos t\rangle dt$

$\vec{F}(\vec{r}(t)) \cdot d\vec{r} = 25\sin^2 t\, dt$

By Stokes' Thm,

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 25 \sin^2 t\, dt$$

$$= 25 \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\, dt$$

$$= \frac{25}{2}\left(t - \tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

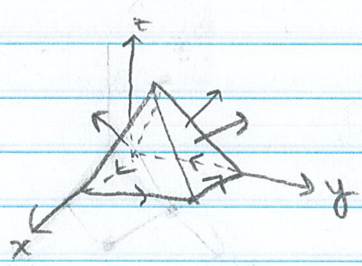
$$= \frac{25}{2}(2\pi - 0 - 0 + 0) = \boxed{25\pi}$$

Suppose $S_1$ and $S_2$ are two oriented surfaces with $\partial S_1 = \partial S_2$ (same boundary curve with same orientation).
By Stokes' Theorem,

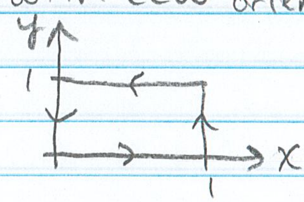
$$\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S_1} \vec{F}\cdot d\vec{r} = \int_{\partial S_2} \vec{F}\cdot d\vec{r} = \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$$

So we can sometimes replace a surface in a surface integral with a nicer surface having the same boundary.

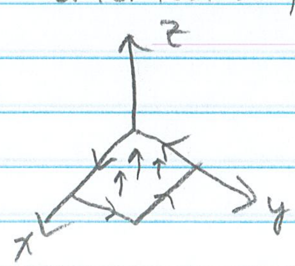
Ex 3. Compute $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle xy, x^2z, yz\rangle$ and $S$ is the pyramid with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(\frac{1}{2},\frac{1}{2},1)$ without the square base, oriented outward.




$\partial S$ is the square in the xy-plane with CCW orientation viewed from above.




Could use $\int_{\partial S} \vec{F}\cdot d\vec{r}$, but would have to use 4 parameterized curves.
Instead, can use the part of the xy-plane enclosed by the square.
By right hand rule, give the plane upward orientation, so $\partial S$ has the induced orientation



The part of the plane can be parameterized as

$\vec{r}(x,y) = \langle x, y, 0 \rangle,\ 0 \le x \le 1,\ 0 \le y \le 1$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{k}$ which is in the correct direction

$\text{Curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2z & yz \end{vmatrix} = (z - x^2)\hat{\imath} - (0-0)\hat{\jmath} + (2xz - x)\hat{k}$

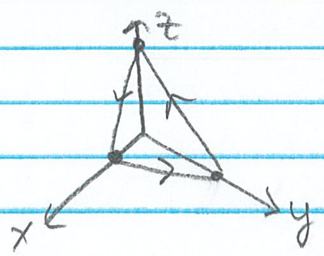
which is parameterized as

$\langle 0 - x^2, 0, 2x \cdot 0 - x \rangle = \langle -x^2, 0, -x \rangle$

$\text{Curl } \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = -x$

$$\iint_{plane} \text{curl } \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 -x \, dx\, dy = \int_0^1 -\tfrac{1}{2}x^2 \Big|_0^1 dy = -\tfrac{1}{2}\int_0^1 dy = \boxed{-\tfrac{1}{2}}$$

Ex 4 Use Stokes' Thm to compute $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \hat{\imath} + (x+yz)\hat{\jmath} + (xy - \sqrt{z})\hat{k}$, $C$ is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant, oriented ccw viewed from above.




Can take $S$ to be the part of the plane $3x+2y+z=1$ in the first octant. By right hand rule, orient $S$ upward to get correct induced orientation on $\partial S = C$.

The plane can be parameterized by

$\vec{r}(x,y) = \langle x, y, 1 - 3x - 2y \rangle,\ 0 \le y \le \frac{1}{2} - \frac{3}{2}x,\ 0 \le x \le \frac{1}{3}$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ (notice this is the normal vector of the plane)

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = (x-y)\hat{\imath} - (y-0)\hat{\jmath} + (1-0)\hat{k}$

$= \langle x-y, -y, 1 \rangle$

$\text{curl}\,\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = 3x - 3y - 2y + 1 = 3x - 5y + 1$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_0^{\frac{1}{3}} \int_0^{\frac{1}{2}-\frac{3}{2}x} (3x - 5y + 1)\, dy\, dx$$

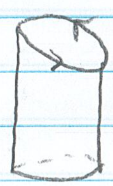
$$= \int_0^{\frac{1}{3}} \left(3xy - \frac{5}{2}y^2 + y\right)\Big|_0^{\frac{1}{2}-\frac{3}{2}x} dx$$

$$= \int_0^{\frac{1}{3}} \left(\frac{3}{2}x - \frac{9}{2}x^2 - \frac{5}{2}\left(\frac{1}{4} - \frac{3}{2}x + \frac{9}{4}x^2\right) + \frac{1}{2} - \frac{3}{2}x\right) dx$$

$$= \int_0^{1/3} \left(-\frac{81}{8}x^2 + \frac{15}{4}x - \frac{1}{8}\right) dx$$

$$= -\frac{27}{8}x^3 + \frac{15}{8}x^2 - \frac{1}{8}x \Big|_0^{1/3} = -\frac{1}{8} + \frac{5}{24} - \frac{1}{24} = \boxed{\frac{1}{24}}$$

Ex 5. Use Stokes' Thm to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = x^2 z\,\hat{\imath} + xy^2\,\hat{\jmath} + z^2\,\hat{k}$, C is the intersection of the plane $x+y+z=2$ and the cylinder $x^2+y^2=4$, oriented ccw as viewed from above.



$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ for $C = \partial S$

Can take S to be the plane $x+y+z=2$ oriented upward.

$\vec{r}_x \times \vec{r}_z = \langle 1, 1, 1 \rangle$ (normal vector of plane)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix} = 0\hat{i} + x^2\hat{j} + y^2\hat{k}$$

$\text{curl}\,\vec{F} \cdot (\vec{r_x} \times \vec{r_y}) = x^2 + y^2$

$\iint_D (x^2+y^2)\,dA$, where $D$ is given by $0 \le x^2+y^2 \le 4$

Use polar:

$$\int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = 2\pi \cdot \tfrac{1}{4}r^4\Big|_0^2 = \boxed{8\pi}$$

Stokes' Thm proves the result that $\text{curl}\,\vec{F} = \vec{0}$ implies $\vec{F}$ is conservative.

Let $C$ be any closed curve, and let $S$ be a surface with $\partial S = C$. By Stokes' Thm,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint \vec{0}\cdot d\vec{S} = 0$$

So $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves $C$, which is equivalent to $\vec{F}$ being conservative.

Green's Thm is a special case of Stokes' Thm. Let $C$ be any positively oriented closed curve in the xy-plane. Take $S$ to be the part of the xy-plane bounded by $C$, with upward orientation. The normal vector for the xy-plane is simply $\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \text{curl}\,\vec{F}\cdot\hat{k}\,dS$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$