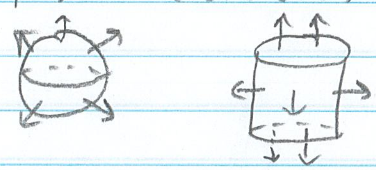
MA 261 - Lesson 26

The Divergence Theorem (16.9)

The divergence theorem applies to closed surfaces with positive (outward) orientation.



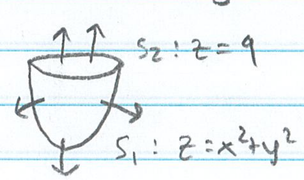
The Divergence Theorem Let S be a closed surface with positive orientation, and let E be the region enclosed by S. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

Again, this is an analogue to the Fundamental Theorem of Calculus, since the integral of a derivative over a region (RHS) is the same as knowing about the boundary of that region.

If S is negatively oriented, simply multiply the whole integral by -1.

Ex 1. Verify the Divergence Theorem is true for $\vec{F}(x,y,z) = y^2z^3\,\hat{\imath} + 2yz\,\hat{\jmath} + 4z^2\,\hat{k}$, over E, the solid enclosed by the paraboloid $z = x^2+y^2$ and the plane $z=9$.



We first compute the surface integral by definition.

(Note: we cannot apply Stokes' Thm since $\vec{F}$ is not the curl of some vector field)

We parameterize S_1 as $\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$, $0 \le x^2+y^2 \le 9$

We should convert to cylindrical coordinates to get a better bound.

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\hat{\imath} - (2r^2\sin\theta)\hat{\jmath} + (r)\hat{k}$$

The hyperboloid must be oriented downward so use

$\langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle$

$$\vec{F}(\vec{r}(r,\theta)) = [(r\sin\theta)^2(r^2)^3]\hat{\imath} + [2(r\sin\theta)(r^2)]\hat{\jmath} + [4(r^2)^2]\hat{k}$$
$$= \langle r^8\sin^2\theta, 2r^3\sin\theta, 4r^4 \rangle$$

$$\vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_\theta \times \vec{r}_r) = 2r^{10}\sin^2\theta\cos\theta + 4r^5\sin^2\theta - 4r^5$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \int_0^{2\pi}\int_0^3 (2r^{10}\sin^2\theta\cos\theta + 4r^5\sin^2\theta - 4r^5)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{2}{11}r^{11}\sin^2\theta\cos\theta + \tfrac{2}{3}r^6\sin^2\theta - \tfrac{2}{3}r^6\right)\Big|_0^3 d\theta$$
$$= \int_0^{2\pi} \left(\frac{2\cdot 3^{11}}{11}\sin^2\theta\cos\theta + 2\cdot 3^5\sin^2\theta - 2\cdot 3^5\right) d\theta$$
$$= \int_0^{2\pi} \frac{2\cdot 3^{11}}{11}\sin^2\theta\cos\theta\,d\theta + \int_0^{2\pi} 3^5(1-\cos 2\theta)\,d\theta - 2\cdot 3^5\int_0^{2\pi} d\theta$$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

$$= \int_0^0 \frac{2\cdot 3^{11}}{11}u^2\,du + 3^5\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} - 2\cdot 3^5\cdot 2\pi$$
$$= 0 + 3^5(2\pi - 0 - 0 + 0) - 2\cdot 3^5\cdot 2\pi$$
$$= 486\pi - 972\pi = -486\pi$$

We parameterize S_2 as $\vec{r}(x,y) = \langle x, y, 9 \rangle$, $0 \le x^2+y^2 \le 9$

We convert to cylindrical coordinates for better bounds

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 9 \rangle$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

The plane $z=9$ has normal vector $\langle 0,0,r \rangle$.

$\vec{F} \cdot \langle 0,0,1 \rangle = 4z^2 = 4(9)^2 = 324$

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \int_0^{2\pi}\int_0^3 324r\,dr\,d\theta$$
$$= 2\pi \cdot 162r^2\Big|_0^3$$
$$= 2\pi \cdot 162 \cdot 3^2 = 2916\pi$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S}_1 + \iint_{S_2} \vec{F} \cdot d\vec{S}_2 = -486\pi + 2916\pi$$
$$= 2430\pi$$

Now, we apply the divergence thm.

$\text{div}\,\vec{F} = 0 + 2z + 8z = 10z$

$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV = \iiint_E 10z\, dV$

Easiest to set up in cylindrical coordinates

$\int_0^{2\pi}\int_0^3\int_{r^2}^9 10zr\, dz\, dr\, d\theta$

$= 2\pi \int_0^3 5z^2 r \Big|_{r^2}^9 dr$

$= 2\pi \int_0^3 (405r - 5r^5)\, dr$

$= 2\pi \left(\frac{405}{2}r^2 - \frac{5}{6}r^6\right)\Big|_0^3$

$= 2\pi \left(\frac{405\cdot 9}{2} - \frac{5\cdot 3^5}{2}\right)$

$= \pi(2430) = \boxed{2430\pi}$

As we can see, The Divergence Thm can be much easier to use than the definition, provided you have a closed surface.

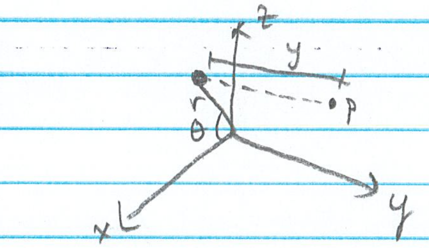
Ex 2. Compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = ye^z\hat{\imath} + 3x^2y\,\hat{\jmath} + z^3\hat{k}$, and S is the positively oriented surface of the solid bounded by the cylinder $x^2+z^2=9$ and the planes $y=-2$ and $y=4$

By the Divergence Thm,

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV$$

$\text{div}\,\vec{F} = 0 + 3x^2 + 3z^2 = 3x^2 + 3z^2$

The region is easiest set up in cylindrical coordinates, but one with y, r, and θ.

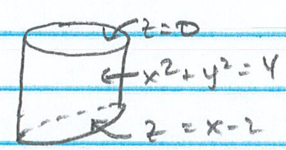


So $x = r\cos\theta$
$y = y$
$z = r\sin\theta$, $x^2+z^2=r^2$

$$\int_0^{2\pi}\int_0^3\int_{-2}^4 3r^2\cdot r \, dy\, dr\, d\theta$$
$$= 2\pi \cdot \int_0^3 3r^3 y \Big|_{y=-2}^{y=4} dr$$
$$= 2\pi \int_0^3 (12r^3+6r^3)\, dr$$
$$= 2\pi \left(\tfrac{9}{2}r^4\right)\Big|_0^3$$
$$= 2\pi\left(\tfrac{729}{2}\right) = \boxed{729\pi}$$

Ex 3. Compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = (x^2+y^2)\hat{\imath} + (xy+2yz)\hat{\jmath} + (xy-z^2)\hat{k}$, and S is the positively oriented surface of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=x-2$ and $z=0$.

Notice: when $x^2+y^2\le 4$, $x\le 2$, so the plane $z=x-2$ is below the plane $z=0$



By the Divergence Theorem, $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

$\operatorname{div}\vec{F} = 2x + x + 2z - 2z = 3x = 3r\cos\theta$ (in cylindrical)

$$\int_0^{2\pi}\int_0^2\int_{r\cos\theta-2}^{0} 3r\cos\theta\cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 3r^2\cos\theta \, z \Big|_{z=r\cos\theta-2}^{z=0} \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (-3r^3\cos^2\theta + 6r^2\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(-\tfrac{3}{4}r^4\cos^2\theta + 2r^3\cos\theta\right)\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} (-12\cos^2\theta + 16\cos\theta)\, d\theta$$

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$= \int_0^{2\pi} (-6 - 6\cos 2\theta + 16\cos\theta)\, d\theta$$

$$= -6\theta - 3\sin 2\theta + 16\sin\theta \Big|_0^{2\pi}$$

$$= (-12\pi - 0 + 0) - (0 - 0 + 0)$$

$$= \boxed{-12\pi}$$